

NEW D1-D5-P GEOMETRIES FROM STRING AMPLITUDES

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ABSTRACT

We derive the long range supergravity fields sourced by a D1-D5-P bound state from disk amplitudes for massless closed string emission. We suggest that since the parameter controlling the string perturbation expansion for this calculation decreases with distance from the bound state, the resulting asymptotic fields are valid even in the regime of parameters in which there is a classical black hole solution with the same charges. The supergravity fields differ from the black hole solution by multipole moments and are more general than those contained within known classes of solutions in the literature, whilst still preserving four supersymmetries. Our results support the conjecture that the black hole solution should be interpreted as a coarse-grained description rather than an exact description of the gravitational field sourced by D1-D5-P bound states in this regime of parameters.

1 Introduction

Type IIB string theory compactified on $S^1 \times \mathcal{M}_4$ (where \mathcal{M}_4 can be either T^4 or K_3) contains a large degeneracy of configurations preserving 4 of the 32 supercharges of the trivial vacuum. Since the seminal papers [1, 2] we know that at zero string coupling $g_s = 0$, the number of these configurations matches the Bekenstein-Hawking entropy of the extremal three-charge black hole. It is expected that a non-renormalization theorem protects this degeneracy from corrections as g_s is switched on, explaining the agreement between the number of string/D-brane configurations at zero coupling and the entropy calculated in the black hole description. In this gravitational regime, g_s is non-zero but can be as small as we like as long as the supergravity charges are large; for instance, this requires that $g_s N > 1$, where N is any one of the D-brane charges of the configuration.

The microscopic derivation of the Bekenstein-Hawking entropy made this class of extremal black holes an ideal arena for trying to address other crucial questions at the heart of black hole physics. In particular, a line of research advocated by Mathur and collaborators has focused on the problem of studying the geometrical backreaction of individual microscopic D-brane configurations (microstates) as the parameter $g_s N$ is increased from zero to a large value (see the reviews [3, 4, 5, 6, 7]). The central question is to understand whether the different elementary configurations produce distinguishable gravitational backgrounds and, if so, to determine the scale at which the differences start being relevant.

This question is closely tied with the information paradox: the Hawking emission process resulting from a classical black hole metric coupled to quantum fields leads to a breakdown of unitarity [8, 9] so if the physical black holes we observe in Nature are to be accurately described by quantum mechanics, then one requires a more refined description of physics at the horizon than that provided by Hawking's description (for recent progress in this area see [10, 11, 12]). One way to avoid the conclusions of the Hawking theorem is to seek a more refined description of the gravitational field sourced by physical black holes; one of the crucial features of the fuzzball proposal [3, 4] is that sizable deviations from the 'naive' black hole geometry appear at a scale proportional to $g_s N$.

One way to test this proposal is to look for solutions of type IIB supergravity which preserve four supersymmetry generators and have the same D1, D5 and Kaluza-Klein charges of the usual black hole, but which differ from the 'naive' black hole geometry at a large scale $\sim g_s N$ (for a review, see [13]). This program builds on the successful analysis of the two-charge D1-D5 configurations, in which case the Kaluza-Klein momentum charge is set to zero. By dualizing the supergravity solutions for a fundamental string with a travelling wave [14, 15], a large class of

horizon-less geometries was found and studied [16, 17] which led to the fuzzball proposal and many further studies, see e.g. [18, 19]. In the D1-D5 duality frame these solutions are everywhere smooth and free of brane sources, a feature which is however duality-frame dependent.

The analysis of the three-charge D1-D5-P configurations has proved much more challenging. A large class of 1/8-BPS smooth and horizon-less supergravity solutions is known [20, 21, 22, 23, 24, 25, 26, 27], however quantizing the degeneracy of these geometries does not yield an entropy of the same order as the entropy of the black hole [28]. Another complication is that the precise relation between the known supergravity solutions and the microstates is less clear than in the two-charge case [29]. One way to match the geometrical and the microscopic descriptions is to use the AdS/CFT dictionary; see for instance [30, 31, 32] for an explicit implementation of this approach (see also [33] for recent progress in understanding the AdS/CFT dictionary in different phases of this system). The basic idea is the following: if a solution is indeed dual to the elementary states of the extremal three-charge black hole, then it should have a “near-horizon” limit, where its asymptotic geometry is $\text{AdS}_3 \times S^3 \times \mathcal{M}_4$; in this limit one can use the AdS/CFT correspondence to read from the geometrical data the corresponding state in the CFT description. However it is more difficult to reverse the logic and use this approach to *construct* new supergravity solutions starting from the definition of a CFT microstate.

In this paper, we use a different approach to derive the large-distance backreacted geometry from a microscopic configuration of D-branes. We exploit the fundamental definition of D-branes as space-time defects which introduce borders in the string world-sheet and identify the left and the right moving excitations of closed strings. By calculating string amplitudes describing the emission of each massless closed string field from world-sheets with disk topology one can derive the large distance fall-off of the various supergravity fields sourced by a given D-brane bound state. This was originally done for the simplest case of 1/2-BPS configurations [34, 35]. Of course, the leading long-range behaviour of a solution is determined by its charges and, if one repeats the same calculation for a simple superposition of D1 and D5-branes, then only the ‘naive’ D1/D5 supergravity solution at large distances is reproduced and, as expected, no higher multipole moments appear.

The backgrounds we are interested in are however more complicated and should correspond to states in the Higgs branch of the D1/D5 world-volume CFT. While a precise quantum description of the states in the Higgs branch is difficult, semiclassically we can characterize them by giving a non-zero vacuum expectation value (vev) to the massless fields in the spectrum of the open string stretched between the D1 and the D5-branes. Even this is quite challenging, as, at the CFT level, the vertex operators for the open strings stretched between different D-branes contain com-

plicated boundary changing operators known as twist/spin fields, as we describe in Section 3.5. However the large-distance expansion of the corresponding backreacted geometry corresponds *order by order* to the standard open string perturbative expansion, where also the open string vevs are treated perturbatively, something which can be done quite straightforwardly as long as we do not have to deal with too many twist fields. Recently it was shown [36] that this approach captures precisely the first non-trivial moments of the two-charge geometries in the D1/D5 duality frame.

The reason for this correspondence lies in how the superghost charge is saturated in superstring perturbation theory. As summarized in Section 4, the main space-time implication of the world-sheet constraint following from the superghost correlators is that the expansion parameter in the perturbative calculation of the geometry produced by each D-brane configuration is $g_s N \alpha' / r^2$, where r is the radial distance in \mathbb{R}^4 at which we are probing the background. So at large enough distances the results derived from string amplitudes should be reliable also in the black hole regime, where $g_s N$ and the Higgs vevs are large.

In this paper we combine the D1/D5 setup of [36] with the microscopic description of a null-wave on D-branes [37, 38, 39, 40] to provide a description of the three-charge microstate at least at the semiclassical level. It was shown in [41] that the boundary state for a D-brane with a travelling wave can be used to derive the large distance behaviour of the two-charge configuration in the D-brane/momentum duality frame. Thus, at the microscopic level, the basic building block for the most general three-charge configuration is provided by a set of D1 and D5-branes, each one with an a priori different wave profile, and a non-trivial vev for the open strings stretched between the two types of D-branes.

We find that the world-sheet analysis of this generic building block is rather non-standard, as the sector of the D1/D5 open strings is described by a logarithmic CFT [42] (for related work see e.g. [43, 44, 45, 46]). We focus our attention in this paper to the simplest case where the wave profiles on the D1 and D5-branes are identical and the worldsheet description is given in terms of a standard CFT¹.

We calculate for these configurations the one-point functions of the massless supergravity fields with disk topologies, and are particularly interested in the contributions which vanish in all two-charge limits, which we describe as ‘*new*’ contributions. We derive the new part of the large-distance expansion of the backreacted geometry up to order $1/r^4$. There are other terms in the $1/r$ expansion of the solution that follow from the non-linearity of gravity, rather than the non-trivial structure of the microstates; in our string approach these contributions are related to world-sheet surfaces with more than one border. We will not try to derive these terms from

¹This class of microstates was recently studied at the level of the probe-brane approximation in [47].

string amplitudes, as it is much easier to do so by solving the standard supergravity equations and by using the results from the disk amplitudes as boundary conditions.

Using the type IIB supersymmetry equations we explicitly check that the supergravity backgrounds derived from these string amplitudes preserve four supercharges. A surprising result is that we find supergravity fields not contained within known classes of 1/8-BPS solutions in the literature [13]: in particular the NS-NS 2-form and the R-R 0 and 4-form potentials are non-trivial, while the metric of the \mathbb{R}^4 part is still hyper-Kahler, apart from the presence of a warp factor.

Our calculation supports the conjecture that individual states should have backreactions with non-trivial multipole moments, whilst an appropriate thermodynamic ensemble average would average out these moments to zero, obtaining the ‘unique’ classical black hole solution with horizon [3, 48, 49, 50].

The paper is structured as follows. In Section 2, we analyze the type IIB supersymmetry equations perturbatively in $1/r$ to constrain the form of 1/8-BPS geometries. We focus on configurations in which the structure space \mathcal{M}_4 only receives an overall warping from the microstate backreaction. We obtain from supergravity a set of conditions which we later use to test the consistency of our string derivation. In Section 3, we review the basic ingredients necessary for the string computation and show why the case of identical profiles on the D1 and D5 branes has a standard CFT description. In Section 4 we calculate the disk one-point functions for the massless supergravity fields and in Section 5 we derive from this data the large-distance behaviour of the supergravity background corresponding to each microscopic D-brane configuration. Finally, in Section 6, we present our conclusions discussing limitations and possible generalizations of this approach for extracting information about the backreactions of D-brane microstates from string amplitudes. The appendices contain technical details of the type IIB supersymmetry equations and the string vertices used in the main text.

2 Supersymmetry analysis

The geometry sourced by a D1-D5-P bound state must preserve the same four supersymmetries as the ‘naive’ black hole with the same charges. We derive in this section, from the supergravity point of view, the constraints imposed on the geometry by the existence of these four conserved supercharges. Though we do not input supersymmetry explicitly in computing string amplitudes, the string results should be compatible with these supersymmetry constraints. This will thus provide a useful and non-trivial check on the string amplitude computation. For the purpose of comparison with the world-sheet results it is sufficient to restrict the supergravity

analysis to $1/r^4$ order in the asymptotic expansion. We have checked that the analysis can be extended to all orders in $1/r$, but we will leave the details of the exact solution for a future work.

2.1 The ansatz

We consider a general ansatz for a IIB configuration compactified on $T^4 \times S^1$ which does not break the isometries along the T^4 directions. For the NSNS fields, a generic ansatz with this property is (in the string frame):

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{Z_1 Z_2}} \left[-\frac{1}{Z_3} d\hat{t}^2 + Z_3 d\hat{y}^2 \right] + \sqrt{Z_1 Z_2} ds_4^2 + \sqrt{\frac{Z_1}{Z_2}} ds_{T^4}^2, \\ B &= -b_0 d\hat{t} \wedge d\hat{y} + b_1 \wedge d\hat{y} + \tilde{b}_1 \wedge d\hat{t} + b_2, \\ e^{2\phi} &= D, \end{aligned} \tag{2.1}$$

where we have introduced the short-hand notation

$$d\hat{t} = dt + k, \quad d\hat{y} = dy + dt - \frac{dt + k}{Z_3} + a_3. \tag{2.2}$$

Here ds_4^2 is a generic Euclidean metric on \mathbb{R}^4 , $ds_{T^4}^2$ is the flat metric on T^4 (which we take to be $ds_{T^4}^2 = \delta_{ab} dz^a dz^b$), Z_1, Z_2, Z_3, b_0, D are 0-forms, k, a_3, b_1, \tilde{b}_1 1-forms, and b_2 a 2-form on \mathbb{R}^4 . Similarly we can write for the RR 0-form and 2-form

$$\begin{aligned} C^{(0)} &= c, \\ C^{(2)} &= -\frac{1}{\tilde{Z}_1} d\hat{t} \wedge d\hat{y} + a_1 \wedge d\hat{y} + \tilde{a}_1 \wedge d\hat{t} + \tilde{\gamma}_2, \end{aligned} \tag{2.3}$$

with c, \tilde{Z}_1 0-forms, a_1, \tilde{a}_1 1-forms and $\tilde{\gamma}_2$ a 2-form on \mathbb{R}^4 . The RR 4-form is constrained to have a self-dual field strength $F^{(5)} = *F^{(5)}$, hence one can write

$$\begin{aligned} F^{(5)} &= f_1 d\hat{t} \wedge dz^4 + f_2 d\hat{y} \wedge dz^4 + g \wedge dz^4 \\ &\quad + Z_2^2 Z_3 f_1 d\hat{y} \wedge dx^4 + \frac{Z_2^2}{Z_3} f_2 d\hat{t} \wedge dx^4 + \frac{Z_2}{Z_1} *_4 g \wedge d\hat{t} \wedge d\hat{y}, \end{aligned} \tag{2.4}$$

where f_1, f_2 are 0-forms, g a 1-form on \mathbb{R}^4 , and dz^4 and dx^4 are the volume forms of T^4 and \mathbb{R}^4 .

At order $1/r^2$ the solution should reduce to the ‘naive’ D1-D5-P black hole; we will moreover use some foresight from the string computation to assert that the 1-forms $k, a_1, \tilde{a}_1, a_3, b_1, \tilde{b}_1$ receive non-trivial contributions first at order $1/r^3$ and that the quantities c, f_1, f_2, g, b_0, b_2 and the metric ds_4^2 do not have non-trivial terms until order $1/r^4$.

Under these assumptions, and discarding terms of order higher than $1/r^4$, the equations of motion for B and $C^{(2)}$ can be approximated by $d * B = 0$, $d * C^{(2)} = 0$, and imply, in particular, that

$$db_2 = *_4 d\tilde{b}_0, \quad d\tilde{\gamma}_2 = *_4 d\tilde{Z}_2, \quad (2.5)$$

for some 0-forms \tilde{b}_0 and \tilde{Z}_2 . Moreover the Bianchi identity $dF^{(5)} = 0$ implies $df_1 = df_2 = 0$, so that one has $f_1 = f_2 = 0$, and $dg = d *_4 g = 0$, so that $g = df$ and $d *_4 df = 0$, for some 0-form f . Then $F^{(5)}$ can be simplified to

$$F^{(5)} = df \wedge dz^4 + *_4 df \wedge d\hat{t} \wedge d\hat{y}. \quad (2.6)$$

We also know that in the ‘naive’ black hole geometry $Z_1 = \tilde{Z}_1$, $Z_2 = \tilde{Z}_2$, $D = Z_1/Z_2$, and we allow these identities to be modified at order $1/r^4$.

In summary our ansatz is given by (2.1), (2.2), (2.3), (2.5), (2.6), with the asymptotic boundary conditions

$$\begin{aligned} Z_1, Z_2, Z_3 &= 1 + O(r^{-2}), \quad \tilde{Z}_i = Z_i + O(r^{-4}) \quad (i = 1, 2), \quad D = \frac{Z_1}{Z_2} + O(r^{-4}), \\ b_0, \tilde{b}_0, c, f &= O(r^{-4}), \quad ds_4^2 = dx_i dx_i + O(r^{-4}), \quad k, a_1, \tilde{a}_1, a_3, b_1, \tilde{b}_1 = O(r^{-3}). \end{aligned} \quad (2.7)$$

2.2 Results

It is a straightforward though quite lengthy exercise to impose the vanishing of the dilatino and gravitino supersymmetry variations and derive the condition this imposes on the various metric coefficients². Some details of this computation, up to order $1/r^4$, are given in Appendix A. The results are

$$\begin{aligned} \tilde{Z}_1 &= Z_1, \quad \tilde{Z}_2 = Z_2, \quad D = \frac{Z_1}{Z_2}, \quad b_0 = \tilde{b}_0 = c = f, \\ \tilde{a}_1 &= a_1, \quad \tilde{b}_1 = b_1, \quad da_1 = *_4 da_1, \quad da_3 = *_4 da_3, \quad db_1 = *_4 db_1, \\ \overline{R}_{ij,kl}^{(4)} &= \frac{1}{2} \epsilon_{ijrs} \overline{R}_{rs,kl}^{(4)}, \end{aligned} \quad (2.8)$$

where $\overline{R}_{ij,kl}^{(4)}$ is the Riemann tensor of ds_4^2 . The last condition is equivalent to require that ds_4^2 be hyper-Kähler.

Moreover the gauge field equations of motion and the ty component of Einstein’s equations imply, at this order, that

$$d *_4 dZ_1 = 0, \quad d *_4 dZ_2 = 0, \quad d *_4 dZ_3 = 0, \quad d *_4 db_0 = 0, \quad d *_4 dk = 0. \quad (2.9)$$

²Though there are many papers that study the supersymmetry conditions for 5D minimal supergravity coupled to various vector and hypermultiplets, as far as we understand none of those results directly apply to the case specified by our ansatz.

Conditions (2.8) and (2.9) are enough to guarantee that all other components of Einstein's equations be satisfied.

3 String world-sheet setup

3.1 D-brane configuration

We consider type IIB string theory on $\mathbb{R}^{1,4} \times S^1 \times T^4$. We denote the 10D coordinates (x^μ, ψ^μ) by $\mu, \nu = t, y, 1, \dots, 8$. We use (i, j, \dots) and x^1, \dots, x^4 for the \mathbb{R}^4 directions, we use (a, b, \dots) and x^5, \dots, x^8 for the T^4 directions and we use (I, J, \dots) to refer to the combined $\mathbb{R}^{1,4} \times S^1$ directions. We work in the light-cone coordinates

$$v = (t + y), \quad u = (t - y) \quad (3.1)$$

constructed from the time and S^1 directions. In Appendix B we collect our CFT conventions, including the form of the BRST charge and some details on the closed string vertices, while in Appendix C we record our conventions for spinors.

We focus on 1/8-BPS configurations composed of D1 and D5-branes with a non-trivial wave. The branes have common Neumann (Dirichlet) boundary conditions along the directions t, y (x^i in the \mathbb{R}^4), while they have mixed Neumann/Dirichlet boundary conditions in the T^4 . We can summarize the D-brane configuration under study in the following table:

$$\begin{array}{c} D1 \\ D5 \end{array} \begin{array}{c|c|c|c} v & u & \mathbb{R}^4 & T^4 \\ \hline \text{x} & \text{x} & f_i^{\text{D1}}(v) & f_a^{\text{D1}}(v) \\ \hline \text{x} & \text{x} & f_i^{\text{D5}}(v) & \text{x} \end{array} \quad (3.2)$$

where “x” denotes a Neumann direction and f indicates the (v -dependent) position of the D-brane in the Dirichlet directions. We will ignore from the very beginning the profile along the T^4 by setting $f_a^{\text{D1}} = 0$. Initially we will allow for independent wave profiles f^{D1} and f^{D5} , before focusing our calculations on the case in which the two profiles are identical.

3.2 Boundary conditions for 1-1 and 5-5 strings

We now review the boundary conditions for an open string with both endpoints on a D-brane carrying a travelling wave. Encoding the effect of the D-brane profile has the effect of resumming all the open string insertions of the vertex

$$V_f = \int \left(\frac{1}{2\alpha'} f_j \partial X^j + \dot{f}_j \psi^j \psi^v \right) dz \quad (3.3)$$

describing the KK charge of the D-brane configurations (see e.g. [51, 52]) so our results will be exact in this respect.

The boundary conditions on the worldsheet fields in the open string picture may be expressed in terms of a reflection matrix R as

$$\tilde{\psi}^\mu = \eta R^\mu_\nu(V) \psi^\nu \quad (3.4)$$

$$\bar{\partial} X^\mu = R^\mu_\nu(V) \partial X^\nu - \delta^\mu_u 8\alpha' \ddot{f}_j \psi^j \psi^u \quad (3.5)$$

where we use a capital V to indicate the string field corresponding to the coordinate v . The parameter η can be set to 1 at $\sigma = 0$, while at $\sigma = \pi$ we have $\eta = 1$ or $\eta = -1$ corresponding to the NS and R sectors respectively.

For 1-1 strings, the holomorphic and the anti-holomorphic world-sheet fields are identified with the reflection matrix $R = R_{D1}$ where (see [41] and references within)

$$(R_{D1})^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4|\dot{f}^{D1}(V)|^2 & 1 & -4\dot{f}_i^{D1}(V) & 0 \\ 2\dot{f}_i^{D1}(V) & 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 & -\mathbb{1} \end{pmatrix}, \quad (3.6)$$

where $\mathbb{1}$ denotes the four-dimensional unit matrix and the indices follow the ordering (v, u, i, a) .

Similarly, for 5-5 strings ending on a D5-brane with profile f^{D5} , the right and left-moving world-sheet fields are identified with R_{D5} , where

$$(R_{D5})^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4|\dot{f}^{D5}(V)|^2 & 1 & -4\dot{f}_i^{D5}(V) & 0 \\ 2\dot{f}_i^{D5}(V) & 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{pmatrix}. \quad (3.7)$$

We note that the reflection matrices R_{D1} and R_{D5} preserve the Minkowski metric

$$R_{D1,D5}^T \eta R_{D1,D5} = \eta. \quad (3.8)$$

Importantly, this setup differs from the case of D-branes at angles or with a constant magnetic field in that our reflection matrices contain a non-trivial function of the coordinate V . Thus, when we use the identification (3.4) on the antiholomorphic part $\tilde{T}_{\tilde{\psi}}$ of the fermionic stress energy tensor (see (B.5) for our conventions), we obtain a new term involving \ddot{f} . In the full stress energy tensor, this is cancelled by a similar term in \tilde{T}_X coming from the non-linear part of the bosonic identification (3.5). So all terms involving \ddot{f} cancel and, thanks to (3.8), we find that our boundary conditions define the usual open string Virasoro algebra.

3.3 Boundary conditions for 1-5 and 5-1 strings

For a string with one endpoint on a D1-brane and one endpoint on a D5-brane, the situation is more complicated. Boundary conditions for a string with endpoints on different D-branes are discussed in [53], from which we now review some relevant expressions. Denoting the bosonic string coordinates by x^μ , the boundary conditions at the two endpoints of the string may be written as

$$\bar{\partial}x^\mu \Big|_{\sigma=0,\pi} = (R_\sigma)^\mu{}_\nu \partial x^\nu \Big|_{\sigma=0,\pi} . \quad (3.9)$$

For a string with both endpoints on the same D-brane, we have $R_0 = R_\pi$ and one may solve the boundary conditions by writing x^μ in terms of a holomorphic field $X^\mu(z)$:

$$x^\mu(z, \bar{z}) = q^\mu + \frac{1}{2} \left[X^\mu(z) + (R_0)^\mu{}_\nu X^\nu(\bar{z}) \right] . \quad (3.10)$$

For a string with endpoints on different D-branes, we may define multi-valued fields $X^\mu(z)$ and introduce a branch cut in the z -plane just below the negative real axis, such that ∂X^μ has a monodromy written in terms of a “monodromy matrix” M :

$$\partial X^\mu(e^{2\pi i} z) = M^\mu{}_\nu \partial X^\nu(z) , \quad \text{where} \quad M \equiv R_\pi^{-1} R_0 . \quad (3.11)$$

Then the boundary conditions are again solved by (3.10), but now with a multivalued field $X^\mu(z)$. In our case we have

$$R_0 = R_{D1} , \quad R_\pi = R_{D5} . \quad (3.12)$$

Then the monodromy matrix turns out to be only a function of the difference between the two profiles, and has the same form as the monodromy matrix for one profile (with the extra minus sign in the T^4 directions):

$$M^\mu{}_\nu = (R_\pi^{-1} R_0)^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4|\dot{f}^{D5} - \dot{f}^{D1}|^2 & 1 & 4(\dot{f}^{D5} - \dot{f}^{D1})_i & 0 \\ 2(\dot{f}^{D5} - \dot{f}^{D1})_i & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & -\mathbb{1} \end{pmatrix} . \quad (3.13)$$

The monodromy matrix has a similar form to that studied in the context of null orbifolds [54, 55]. This can be seen by defining

$$f_i^- = f_i^{D5} - f_i^{D1} , \quad (3.14)$$

and then by writing

$$M = M_0 \exp \left(2\pi \dot{f}_i^- \mathcal{J}_i \right) , \quad (3.15)$$

where M_0 is the monodromy matrix for $f^- = 0$ (see (3.18)) and

$$(\dot{f}^- \cdot \mathcal{J})^\mu{}_\nu = \frac{1}{\pi} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2\dot{f}_i^- & 0 \\ \dot{f}_i^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \quad (3.16)$$

Thus the monodromy (3.11) for the coordinates ∂X can be written as follows

$$\partial X^\mu(e^{2\pi i} z) = \left[\exp \left(2\pi \dot{f}_j^- \mathcal{J}_j \right) \right]^\mu{}_\nu \partial X^\nu(z), \quad (3.17)$$

The unusual feature is that $\dot{f}^- \mathcal{J}$ is nilpotent and thus not diagonalizable. A consequence of this is that the sector of open strings stretched between D-branes with different profiles has the structure of a logarithmic CFT. In particular, one finds that at each level there is a Jordan block of rank three, related to the property $(\dot{f}^- \mathcal{J})^3 = 0$. We shall postpone the study of the full analysis of this problem to later work, and in this paper we shall treat the simpler scenario in which the profiles are equal, in which case one has $f^- = 0$.

3.4 Equal D1 and D5 profiles

We now specialize to the case in which both the D1 and the D5 branes are wrapped n_w times around y . Letting the length of the y direction be $2\pi R$, each brane then has total length $L_T = 2\pi n_w R$ and we use \hat{v} for the corresponding world-volume coordinate on the D-branes, having periodicity L_T . Moreover we focus on the case in which the D1 and D5 branes have identical profiles, which we denote by $f \equiv f^{D1} = f^{D5}$. Clearly this common profile f^i satisfies $f^i(\hat{v} + L_T) = f^i(\hat{v})$.

We see from the discussion in the previous subsection that in this case the monodromy matrix reduces to that of the two-charge D1-D5 system as studied in [36], i.e.

$$(M_0)^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & -\mathbb{1} \end{pmatrix}. \quad (3.18)$$

In this case the worldsheet CFT is not logarithmic and one can use the boundary conditions (3.6) with $f^{D1} = f$ in the calculation of closed string emission from a D1-D5 disk. This is the approach we follow in Section 4.2.

We will later also require the boundary conditions for the left and right moving spin fields. These are

$$\tilde{S}^{\hat{A}} = (\mathcal{R}_{D1})^{\hat{A}}{}_{\hat{B}} S^{\hat{B}} \quad (3.19)$$

where \mathcal{R}_{D1} is the spinor representation of the reflection matrix R_{D1} . For a flat D1 brane, $\mathcal{R}_{D1} = \Gamma^{ty}$ however for a D-brane with a travelling wave we can read \mathcal{R}_{D1} from the R-R zero mode boundary state for a D1 brane with travelling wave described by the profile f^i . This was calculated in [41] to be

$$|D1; P\rangle_{\psi,0}^{(\eta)} = \mathcal{M}_{AB}^{(\eta)} |A\rangle_{-\frac{1}{2}} |\tilde{B}\rangle_{-\frac{3}{2}} \quad (3.20)$$

where, following our spinor conventions (given in Appendix C) in which the gamma matrices in ten dimensions are denoted $\Gamma_{(10)}^\mu$, we have

$$\mathcal{M}^{(\eta)} = i C \left(\frac{1}{2} \Gamma_{(10)}^{uv} + \dot{f}^i(v) \Gamma_{(10)}^{iv} \right) \left(\frac{1 - i\eta \Gamma_{11}}{1 - i\eta} \right), \quad (3.21)$$

so we read off

$$\mathcal{R}_{D1} = \left(\frac{1}{2} \Gamma_{(10)}^{uv} + \dot{f}^i(v) \Gamma_{(10)}^{iv} \right). \quad (3.22)$$

3.5 Twisted open string vertex operators

In Section 4.2 we calculate amplitudes on a mixed disk with half its boundary on a D1 and the other half on the D5, and two twisted vertex operator insertions as studied in [56, 36], details of which we record here.

Since the monodromy matrix is now given simply by (3.18), we use the same twisted open string vertices considered in [36], which take the form

$$V_\mu = \mu^A e^{-\frac{\varphi}{2}} S_A \Delta, \quad V_{\bar{\mu}} = \bar{\mu}^A e^{-\frac{\varphi}{2}} S_A \Delta \quad (3.23)$$

where μ^A and $\bar{\mu}^A$ are Chan-Paton matrices with $n_1 \times n_5$ and $n_5 \times n_1$ components respectively, S_A are the $SO(1,5)$ spin fields, φ the free boson appearing in the bosonized language of the worldsheet superghost (β, γ) , and Δ is the bosonic twist operator with conformal dimension $\frac{1}{4}$ which acts along the four mixed ND directions (in which the monodromy matrix has the value -1) and changes the boundary conditions from Neumann to Dirichlet and vice versa.

We focus on open string condensates involving only states from the Ramond sector. Notice that states in the Ramond sector will break the $SO(4)$ symmetry of the DD directions \mathbb{R}^4 , while they are invariant under the $SO(4)$ acting on the compact T^4 torus. The most general condensate of Ramond open strings can be written as:

$$\bar{\mu}^A \mu^B = v_I (C\Gamma^I)^{[AB]} + \frac{1}{3!} v_{IJK} (C\Gamma^{IJK})^{(AB)}, \quad (3.24)$$

where the parenthesis on the indices A, B are meant to remind that the first term is automatically antisymmetric, while the second one is symmetric. Thus the open string bispinor condensate is specified by a one-form v_I and an self-dual three-form v_{IJK} . The self-duality of v_{IJK} follows from $\bar{\mu}^A$ and μ^B having definite 6D chirality and can be written as

$$v_{IJK} = \frac{1}{3!} \epsilon_{IJKLMN} v^{LMN}. \quad (3.25)$$

In this paper we shall consider only the components of v_{IJK} which have one leg in the t, y directions and two legs in the \mathbb{R}^4 ; this choice of components was associated

to considering profiles only in the \mathbb{R}^4 directions in [36]. In t, y coordinates with $\epsilon_{1234} = 1$, we see that (3.25) becomes

$$v_{yij} = \frac{1}{2} \epsilon_{ijkl} v_{tkl} . \quad (3.26)$$

The self-duality properties in light-cone coordinates are then

$$v_{uij} = -\frac{1}{2} \epsilon_{ijkl} v_{ukl} , \quad v_{vij} = +\frac{1}{2} \epsilon_{ijkl} v_{vkl} . \quad (3.27)$$

Since the spinors $\bar{\mu}^A$ and μ^B carry $n_5 \times n_1$ and $n_1 \times n_5$ Chan-Paton indices, the condensate $\bar{\mu}^A \mu^B$ must be thought of as the vev for the sum

$$\sum_{m=1}^{n_1} \sum_{n=1}^{n_5} \bar{\mu}_{mn}^A \mu_{nm}^B , \quad (3.28)$$

which, for generic choices of the Chan-Paton factors, is of order $n_1 n_5$.

4 D-brane geometrical backreaction

In this section we derive the geometrical backreaction of the D-brane configuration discussed in the previous section, starting from the couplings of the closed string massless fields. In perturbation theory, these couplings are captured by the one-point functions of each closed string state on world-sheets with boundaries. As we are not interested in purely quantum gravity effects (*i.e.* couplings weighted by the Planck length) we will ignore all contributions to the one-point functions from topologies which have handles and focus only on planar world-sheets with boundaries. The non-trivial vacuum expectation values for the open string fields (3.3) and (3.23) should ensure that we are considering, at least semiclassically, not merely a naive superposition of three charges but a real bound state.

As discussed in the previous section, the boundary conditions (3.4) and (3.5) resum all the open string insertions describing the KK momentum charge of the D-brane configurations, so our results will be exact in this respect. On the contrary we will treat perturbatively the open string insertions (3.23) related to the vev of the strings stretched between the D1 and D5 branes.

The interesting microstates, for which we might expect a gravitational description, have large open string vevs and so, in principle, we should resum amplitudes with many twisted vertices. However, as briefly mentioned in the Introduction, it is possible to check that string amplitudes with a different number of open string insertions (3.23) contribute to different terms in the large distance expansion of the corresponding gravity solution. The argument goes as follows: disk amplitudes

are non vanishing only if the total superghost charge of the correlator is -2 ; the open string vertices (3.23), which must be always paired in order to have a consistent boundary, are in the $-1/2$ picture, so each insertion of $V_\mu, V_{\bar{\mu}}$ in a non-zero correlator requires an extra e^ϕ factor to keep the correlator non-trivial.

Thus the expansion in the number of twisted open string insertions is actually weighted by the e^ϕ charge we need to saturate. This can be done by inserting in the amplitude the supercurrent, or equivalently by changing the picture of the emitted closed string vertex. From the form of the supercurrent (given in (B.6)) we see that each e^ϕ factor is accompanied by a ∂X , which in the amplitudes we are interested in becomes a factor of the closed string momentum k . In configuration space each factor of k becomes a factor of $1/r$, r being the radial coordinate in the \mathbb{R}^4 . Thus amplitudes with m pairs of $V_\mu, V_{\bar{\mu}}$ vertices contribute to the geometric backreaction with terms which decay at least as $1/r^m$ at large distances.

We also have the standard open string loop expansion weighted by the number of borders inserted in the diagrams and we need to justify why we focus on disk amplitudes. The reason we can perform a perturbation expansion in the regime of parameters of interest is that the open string loop expansion parameter for the calculation of the value of the backreacted field at a radial distance r (for a generic Dp -brane) is

$$\epsilon = g_s N \left(\frac{\alpha'}{r^2} \right)^{\frac{7-p}{2}} \quad (4.1)$$

where here N counts the number of D1 or D5 branes. For fixed large $g_s N$, ϵ can be made arbitrarily small by choosing to examine the fields at large enough r .

One can see that the quantity ϵ controls the open string perturbation expansion as follows. Adding an extra border to the string worldsheet gives a factor of $g_s N$ since there are N choices of which D-brane the open string endpoints can end on. It also introduces a loop momentum integral, two extra propagators, and reduces the background superghost charge by two units, requiring us to increase the picture of the vertex operators into a picture two units higher.

Qualitatively, each of these contributes as follows: At large distances, the loop momentum integral is dominated by the closed string channel, effectively resulting in an integral over the Dirichlet directions, $\int d^{9-p}k$. The two propagators bring two factors of $1/k^2$, and the picture-changing procedure brings a factor of k^2 as we have described. Thus all together we have an additional integral of the form

$$\int d^{9-p}k \frac{1}{k^2} \sim \frac{1}{r^{7-p}} \quad (4.2)$$

and so restoring units of α' we indeed find that ϵ is the appropriate dimensionless expansion parameter.

In the following sections we focus on the contributions which depend on all three charges of the microstate being present, and so vanish when any one charge is turned off; we refer to these as the ‘new’ fields. The new fields are not the only contributions appearing up to order $1/r^4$; there are also contributions from diagrams with more borders which contribute at the same order. However as mentioned in the Introduction, amplitudes with many disconnected borders are both more difficult to derive and also less interesting. At large distances, they should simply reproduce the contributions due to the non-linear nature of gravity; these contributions are more easily calculated in the low-energy limit, by solving the supergravity Killing spinor equations (and equations of motion). The reason is that the momentum of each closed string exchanged between a probe placed at large distances and the D-brane bound state is very small. So the contribution to the string amplitudes is dominated by world-sheets which look like tree-level gravity Feynman diagrams where each boundary represents a source. This is a diagrammatic representation for the perturbative solution of the supergravity equations of motion as performed in Section 2.

Thus in the following we focus only on the leading contributions at large distances which are induced by the amplitudes with one border (see Figure 1) and those with one border and one pair of open vertices $V_\mu, V_{\bar\mu}$ (see Figure 2). This should be sufficient to capture, in the supergravity solution corresponding to any microstate, the interesting new terms up to order $1/r^4$.

4.1 Amplitudes with one type of boundary

The most direct way to derive the one-point functions from a disk with only one type of boundary is to use the boundary state formalism [34]. In our case we have two contributions, depending on whether the boundary is ending on the D1 or D5 branes, see Figure 1. These one-point couplings, and the corresponding contributions to the background geometry, were calculated in [41], by using the boundary state for a D-brane with a null wave derived in [38, 39, 40]. Here we summarize the derivation of the NS-NS fields in order to clarify a point on how to separate the dilaton and the metric contribution which will be useful in the following. For the R-R sector, we will just recall the results of [41].

Let us focus on the $D1_f$ diagram Figure 1; the calculation for the other contribution will be completely analogous. We can view the wrapped D1-brane as a collection of n_w different D-brane strands, with a non-trivial holonomy gluing these strands together. Each strand carries a segment $f_{(s)}^i$, with $s = 1, \dots, n_w$ of the full profile. The boundary state describing the wrapped D1-brane can be expanded in terms of

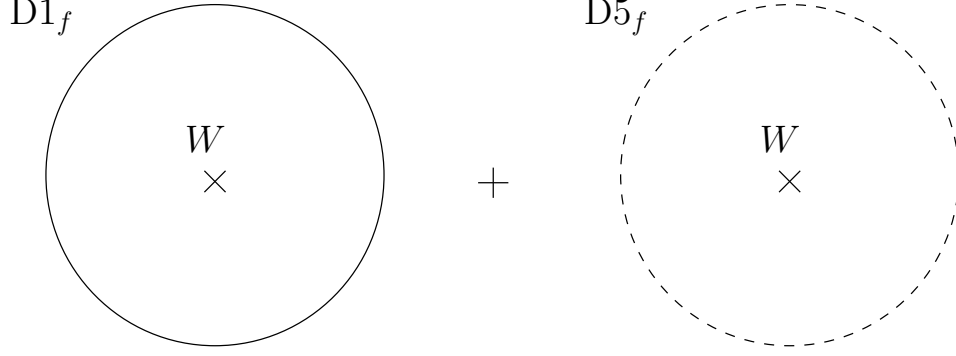


Figure 1: The simplest non-trivial contributions to the one-point function of closed string state W : two disk diagrams where the border lies entirely on the D1 brane (for the first amplitude) or on the D5 brane (for the second amplitude).

the closed string perturbative states. The first terms of this expansions are

$$\begin{aligned}
|D1; f\rangle &= -i \frac{\kappa \tau_1}{2} \sum_{s=1}^{n_w} \int du \int_0^{2\pi R} dv \int \frac{d^4 p_i}{(2\pi)^4} e^{-ip_i f_{(s)}^i(v)} \frac{c_0 + \tilde{c}_0}{2} \\
&\quad c_1 \tilde{c}_1 \left[-\psi_{-\frac{1}{2}}^\mu ({}^t R_{D1})_{\mu\nu} \tilde{\psi}_{-\frac{1}{2}}^\nu + \gamma_{-\frac{1}{2}} \tilde{\beta}_{-\frac{1}{2}} - \beta_{-\frac{1}{2}} \tilde{\gamma}_{-\frac{1}{2}} + \dots \right] |u, v, p_i, 0\rangle_{-1, \tilde{-1}}
\end{aligned} \tag{4.3}$$

where $\tau_1 = [2\pi\alpha'g_s]^{-1}$ is the physical tension of a D1-brane and where ${}^t R_{D1}$ is the transpose of the reflection matrix R_{D1} given in (3.6). The ket in (4.3) represents a closed string state obtained by acting on the $SL(2, C)$ invariant vacuum with a $e^{ip_i x^i}$ in the \mathbb{R}^4 directions. We also wrote the delta functions on the p_u and p_v momenta as integrals in configuration space du, dv . The boundary state enforces the identification (3.4), which in the approximation (4.3) holds just for the first oscillator $\tilde{\psi}_{-1/2}^\mu$.

The second line of (4.3) contains all the massless NS-NS states and we can separate the irreducible contributions by taking the scalar product with each state, details of which are given in Appendix B. The dilaton state has the form (see (B.9) for details):

$$\lim_{z \rightarrow 0} W_{\text{dil}}^{(-2)} |0\rangle = \left(\eta_{\mu\nu} c_1 \psi_{-\frac{1}{2}}^\mu \tilde{c}_1 \tilde{\psi}_{-\frac{1}{2}}^\nu + c_1 \gamma_{-\frac{1}{2}} \tilde{c}_1 \tilde{\beta}_{-\frac{1}{2}} - c_1 \beta_{-\frac{1}{2}} \tilde{c}_1 \tilde{\gamma}_{-\frac{1}{2}} \right) |k\rangle_{-1} |\tilde{k}\rangle_{-1} \tag{4.4}$$

while the graviton and the B-field are given by the symmetric and the antisymmetric parts of \mathcal{G} in $\mathcal{G}_{\mu\nu} c_1 \psi_{-1/2}^\mu \tilde{c}_1 \tilde{\psi}_{-1/2}^\nu |k_i\rangle_{-1, \tilde{-1}}$.

Some care is needed to read the diagonal components of the graviton, as we have to select a state that is orthogonal to (4.4). For these components we use

$$\left[B \left(\eta_{\alpha\beta} c_1 \psi_{-\frac{1}{2}}^\alpha \tilde{c}_1 \tilde{\psi}_{-\frac{1}{2}}^\beta \right) + C \left(c_1 \psi_{-\frac{1}{2}}^i \tilde{c}_1 \tilde{\psi}_{-\frac{1}{2}}^i + c_1 \gamma_{-\frac{1}{2}} \tilde{c}_1 \tilde{\beta}_{-\frac{1}{2}} - c_1 \beta_{-\frac{1}{2}} \tilde{c}_1 \tilde{\gamma}_{-\frac{1}{2}} \right) \right] |k_i\rangle_{-1, \tilde{-1}}$$

where $B = -(7-p)/4$ and $C = (p+1)/4$ and we temporarily use the notation, for a generic D p -brane, that the indices α, β (i) are parallel (transverse) to the brane world-volume. The values of B and C ensure that this state has a vanishing scalar product with (4.4). We note that, if we analytically continue the momentum k_i so as to have $k^2 = 0$, the two round parenthesis are separately BRST-invariant.

We can now separate two types of contributions to the first diagram in Figure 1. The first type involves the diagonal part of the reflection (3.6), while the second one follows from the elements in R_{D1} which depend on f . The result for the diagonal terms is almost identical to the result for a static flat D-brane [34]: the one-point function for the canonically normalized dilaton $\hat{\phi}$ is

$$\mathcal{A}_{\text{dil}}^{\text{D1}}(k) = -i \frac{\kappa \tau_1}{2} V_u \sqrt{2} \hat{\phi} \int_0^{L_T} d\hat{v} e^{-ik \cdot f(\hat{v})}, \quad (4.5)$$

where V_u is the (infinite) volume along the u direction. As we shall see, this term and the analogous D5-P contribution are the only contributions to the dilaton for the D-brane configuration under analysis. Similarly the one-point function for the diagonal components $\hat{h}_{\mu\mu}$ of the canonically normalized graviton is

$$-i \frac{\kappa \tau_1}{2} V_u \int_0^{L_T} d\hat{v} e^{-ik \cdot f(\hat{v})} \left(-\frac{3}{2}(-\hat{h}_{tt} + \hat{h}_{yy}) + \frac{1}{2}(\hat{h}_{ii} + \hat{h}_{aa}) \right). \quad (4.6)$$

Here we already summed the contributions over all strands and so the integrals over v in each strand in (4.3) have been combined in a single integral over \hat{v} extended from 0 to L_T .

The f -dependent terms in the reflection matrix switch on new couplings with the off-diagonal terms of the metric³. By using the expression for the reflection (3.6), one can see that the complete graviton coupling induced by the first diagram in Figure 1 is

$$\begin{aligned} \mathcal{A}_{\text{gra}}^{\text{D1}}(k) = & -i \frac{\kappa \tau_1}{2} V_u \int_0^{L_T} d\hat{v} e^{-ik \cdot f(\hat{v})} \left[-\frac{3}{2}(-\hat{h}_{tt} + \hat{h}_{yy}) \right. \\ & \left. + \frac{1}{2}(\hat{h}_{ii} + \hat{h}_{aa}) - 2\hat{h}_{vv}|f|^2 + 4\hat{h}_{vi}f^i \right]. \end{aligned} \quad (4.7)$$

In the NS-NS sector, the expansion of the boundary state for a D5-brane with a null wave f is completely analogous to the D1 case (4.3), except for the appearance of the D5-brane tension $\tau_5 = [(2\pi\sqrt{\alpha'})^5\sqrt{\alpha'}g_s]^{-1}$ and the reflection matrix R_{D5} given

³As we are considering a non-trivial profile only the \mathbb{R}^4 directions, we do not have any contributions to the B-field.

in (3.7). Thus we can read right away the contribution to NS-NS couplings from the second diagram in Figure 1:

$$\mathcal{A}_{\text{dil}}^{\text{D5}}(k) = i \frac{\kappa \tau_5}{2} V_u V_4 \sqrt{2} \hat{\phi} \int_0^{L_T} d\hat{v} e^{-ik \cdot f(\hat{v})}, \quad (4.8)$$

$$\begin{aligned} \mathcal{A}_{\text{gra}}^{\text{D5}}(k) = & -i \frac{\kappa \tau_5}{2} V_u V_4 \int_0^{L_T} d\hat{v} e^{-ik \cdot f(\hat{v})} \left[-\frac{1}{2} (-\hat{h}_{tt} + \hat{h}_{yy} + \hat{h}_{aa}) \right. \\ & \left. + \frac{3}{2} \hat{h}_{ii} - 2\hat{h}_{vv} |\dot{f}|^2 + 4\hat{h}_{vi} \dot{f}^i \right] \end{aligned} \quad (4.9)$$

where we recall the notation that V_4 is the volume of the compact space.

The details of the R-R calculation may be found in [41], here we just recall the results:

$$\mathcal{A}_{\text{RR}}^{\text{D1}}(k) = -i \sqrt{2} \kappa \tau_1 V_u \int_0^{L_T} d\hat{v} e^{-ik \cdot f(\hat{v})} \left[2\hat{C}_{uv}^{(2)} + \hat{C}_{vi}^{(2)} \dot{f}^i \right] \quad (4.10)$$

$$\mathcal{A}_{\text{RR}}^{\text{D5}}(k) = -i \sqrt{2} \kappa \tau_5 V_u V_4 \int_0^{L_T} d\hat{v} e^{-ik \cdot f(\hat{v})} \left[2\hat{C}_{uv5678}^{(6)} + \hat{C}_{vi5678}^{(6)} \dot{f}^i \right]. \quad (4.11)$$

4.2 Amplitudes with two types of boundary

We next calculate the contribution coming from diagrams with two types of boundary. These diagrams were studied in the case of a D1-D5 bound state without momentum charge in [36]; in this section we introduce momentum charge by using the boundary conditions derived earlier. The world-sheet topology for these amplitudes is depicted in Figure 2 and involves a mixed disk with half its boundary on a D1 and the other half on the D5 brane. Clearly this type of diagram is absent for the naive D1/D5 superposition, where the fields living on the D-brane world-volume are set to zero. On the contrary, the configurations corresponding to D-brane bound states have a non-zero vev for the massless fields in the spectrum of the open strings stretched between the D1 and D5-branes. In our perturbative approach, these vevs are described through the insertion of pairs of vertex operators, such as V_μ , corresponding to an open string stretching from the D1 to the D5-branes, and $V_{\bar{\mu}}$, corresponding to an open string with the opposite orientation. For the microstates we are interested in, these vertex operators are given in (3.23).

Thus the amplitude we need to calculate is

$$\mathcal{A}_{NS,R}^{\text{D1-D5}} = \int \frac{\prod_{i=1}^4 dz_i}{dV_{\text{CKG}}} \left\langle V_\mu(z_1) W_{NS,R}^{(-k)}(z_2, z_3) V_{\bar{\mu}}(z_4) \right\rangle_f, \quad (4.12)$$

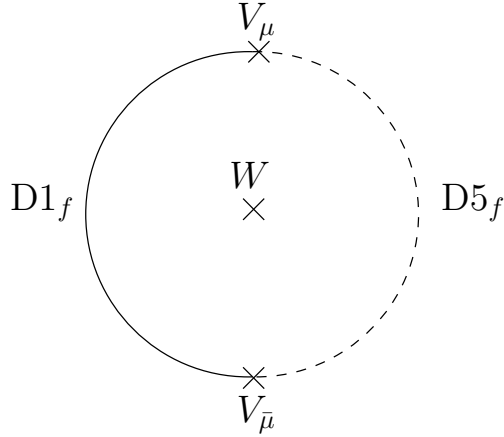


Figure 2: The simplest amplitude involving all three charges of the microstate: the topology of the worldsheet is that of a mixed disk diagram where part of the border lies on the D1 brane and part on the D5 brane.

where the subscript f reminds that, in this disk correlator, the identification between holomorphic and anti-holomorphic components depends on the profile of the D-branes. In the parameterization where the disk is mapped to the upper half of the complex plane, z_1 and z_4 are purely real as they represent the positions of the open string vertices and lie on the boundary of the world-sheet, while $z_2 = \bar{z}_3$ is the position of the closed string vertex in the interior of the world-sheet surface. In order to have a non-trivial correlator we need to saturate the superghost charge of the disk (-2) : the two open string vertices together contribute half of this total charge, so the closed string insertion has to carry globally another -1 superghost charge. Finally in (4.12) we did not keep track of the factors contributing to the overall normalization. This normalization can be reabsorbed in the dictionary between the string condensate (3.24) and the supergravity results and so it is not relevant for the comparison with the ansatz of Section 2.

4.2.1 NS-NS Amplitude

In the NS sector the holomorphic and the anti-holomorphic parts of the vertices have integer superghost charge, and so the constraint on the superghost charge forces us to work in an asymmetric picture; for instance in the $(0, -1)$ picture, the closed string vertex operators representing the emission of graviton or B-field is

$$W_{NS}^{(k)} = \mathcal{G}_{\mu\nu} \left(\partial X_L^\mu - i \frac{k}{2} \cdot \psi \psi^\mu \right) e^{i \frac{k}{2} \cdot X_L(z)} \tilde{\psi}^\nu e^{-\tilde{\varphi}} e^{i \frac{k}{2} \cdot X_R(\bar{z})} + \dots, \quad (4.13)$$

where the dots stand for other terms that ensure the BRST invariance of the vertex, but that do not play any role in the correlator under analysis. For consistency, we should be able to choose the $(-1, 0)$ picture, where the roles of the holomorphic and

the anti-holomorphic parts in (4.13) are swapped, or any linear combination of these two choices. We show in Appendix B that all these choices yield the same result.

The dilaton vertex we need to use is described in Appendix B and is written in (B.10). This vertex has a term involving the (ξ, η) fields in the superghost sector which does not contribute to the correlator (4.12). The other term, which is relevant in our case, has the same structure as the vertex (4.13), but with $\mathcal{G}_{\mu\nu} = \eta_{\mu\nu}$ and a symmetric linear combination of the $(-1, 0)$ and $(0, -1)$ structures. Since both structures yield the same result in the correlator (4.12), we can use the simplified form (4.13) for all massless NS-NS states, where the polarization \mathcal{G} is equal to the Minkowski metric for the dilaton, while it is a symmetric (antisymmetric) tensor for the graviton (B-field).

$SO(1, 5)$ invariance, which is broken only by the boundary conditions, dictates the form of the amplitude (4.12) up to an integral over the world-sheet punctures which was calculated in [36]. By adapting that result to our case, we obtain

$$\mathcal{A}_{\text{NS}}^{\text{D1-D5}} = -2\sqrt{2}\pi V_u \int_0^{L_T} d\hat{v} e^{-ik \cdot f(\hat{v})} k^K \mathcal{G}^{IJ} ({}^t R)_J{}^M v_{IMK} , \quad (4.14)$$

where we take ${}^t R$ to be the transpose of the reflection matrix R_{D1} given in (3.6); we could also use $R = R_{\text{D5}}$ and we would obtain the same result for the correlator under analysis. The f -dependent exponential factor follows from the zero-mode part of the $e^{ikX_{L,R}/2}$ terms in the vertex operator (4.13), as a consequence of the Dirichlet boundary conditions for the string coordinates $x^i = (X_L^i + X_R^i)/2$ in the \mathbb{R}^4 . The integral over \hat{v} follows, as in the previous section, from the zero-mode correlator along the Neumann direction v , and, after combining the contributions from the different D-brane strands, we can write the full amplitude as an integral over the world-volume coordinate \hat{v} .

Expanding the above amplitude (4.14) for the D1-D5 condensate in which only the components v_{uij} and v_{vij} are non-zero and k is only in the \mathbb{R}^4 directions, we obtain

$$\begin{aligned} \mathcal{A}_{\text{NS}}^{\text{D1-D5}} = & 2\sqrt{2}\pi V_u \int_0^{L_T} d\hat{v} e^{-ik \cdot f(\hat{v})} k^l \left[(\mathcal{G}^{uj} + \mathcal{G}^{ju}) v_{ujl} + (\mathcal{G}^{vj} + \mathcal{G}^{jv}) v_{vjl} \right. \\ & \left. + 4\mathcal{G}^{jv} v_{ujl} |\dot{f}(\hat{v})|^2 - 4\mathcal{G}^{ij} v_{uil} \dot{f}^j(\hat{v}) - 2\mathcal{G}^{vv} v_{vil} \dot{f}^i(\hat{v}) - 2\mathcal{G}^{uv} v_{uil} \dot{f}^i(\hat{v}) \right] , \end{aligned} \quad (4.15)$$

where the first line includes the two-charge D1-D5 contribution, and the second one contains the new D1-D5-P contributions, which vanish in each of the two-charge limits, *i.e.* if we set to zero either the v_{IJK} condensate or the profile f^i .

It is interesting to notice that the result (4.15) vanishes if we focus on the emission of a dilaton. As argued above, in our amplitude one can effectively use $\eta_{\mu\nu}$ for the

dilaton polarization. There are two terms in the second line of (4.15) which can potentially contribute to the dilaton amplitude, however, in our case, they cancel each other,

$$4\eta^{ij} v_{uil} \dot{f}^j + 2\eta^{uv} v_{uil} \dot{f}^i = v_{uil} \dot{f}^i (4 - 4) = 0 ,$$

where we used $\eta^{ij} = \delta^{ij}$ and $\eta^{uv} = -2$. Thus the only non-trivial contributions from the diagram in Figure 2 are for the graviton and B-field and can be read from Eq. (4.15) by using

$$\mathcal{G}_{\mu\nu} = \hat{h}_{\mu\nu} + \frac{1}{\sqrt{2}} \hat{b}_{\mu\nu} , \quad (4.16)$$

where, as before, \hat{h} and \hat{b} are the canonically normalized supergravity fields. Thus we obtain the graviton coupling

$$\begin{aligned} \mathcal{A}_{\text{gra}}^{\text{D1-D5}} = & -2\sqrt{2}\pi V_u \int_0^{L_T} d\hat{v} e^{-ik \cdot f(\hat{v})} k^l \left[-2\hat{h}^{uj} v_{ujl} - 2\hat{h}^{vj} v_{vjl} \right. \\ & \left. - 4\hat{h}^{jv} v_{ujl} |\dot{f}(\hat{v})|^2 + 4\hat{h}^{ij} v_{uil} \dot{f}_j(\hat{v}) + 2\hat{h}^{vv} v_{vil} \dot{f}^i(\hat{v}) + 2\hat{h}^{uv} v_{uil} \dot{f}^i(\hat{v}) \right] \end{aligned} \quad (4.17)$$

and the B-field coupling

$$\begin{aligned} \mathcal{A}_{\text{B}}^{\text{D1-D5}} = & -2\pi V_u \int_0^{L_T} d\hat{v} e^{-ik \cdot f(\hat{v})} k^l \\ & \times \left[-4\hat{b}^{jv} v_{ujl} |\dot{f}(\hat{v})|^2 + 4\hat{b}^{ij} v_{uil} \dot{f}_j(\hat{v}) + 2\hat{b}^{uv} v_{uil} \dot{f}^i(\hat{v}) \right] . \end{aligned} \quad (4.18)$$

4.2.2 R-R Amplitude

In the R-R sector the standard form for the massless closed string vertices is, in the $(-1/2, -1/2)$ picture,

$$W_R^{(k)} = \frac{1}{8} \mathcal{F}_{\hat{A}\hat{B}} e^{-\frac{\varphi}{2}} S^{\hat{A}} e^{i\frac{k}{2} \cdot X_L}(z) e^{-\frac{\tilde{\varphi}}{2}} \tilde{S}^{\hat{B}} e^{i\frac{k}{2} \cdot X_R}(\bar{z}) , \quad (4.19)$$

where $\mathcal{F}_{\hat{A}\hat{B}}$ contains the fields strengths F of the R-R fields. It can be expanded on a basis of ten dimensional Gamma matrices and contains a 1, a 3 and a self-dual 5-form

$$\mathcal{F}_{\hat{A}\hat{B}} = \sum_{n=1,3,5} \frac{1}{n!} F_{\mu_1 \dots \mu_n}^{(n)} (C_{10} \Gamma_{(10)}^{\mu_1 \dots \mu_n})_{\hat{A}\hat{B}} . \quad (4.20)$$

The standard relation between the field strength $F^{(n)}$ and its $U(1)$ gauge potentials $C^{(n-1)}$ reads in momentum space as follows

$$F_{I_1 \dots I_n}^{(n)} = n i k_{[I_1} \hat{C}_{I_2 \dots I_n]}^{(n-1)} . \quad (4.21)$$

The amplitude we next calculate is again (4.12), now with the R-R vertex (4.19) inserted. The holomorphic and anti-holomorphic spin fields are identified via the spinor representation of the reflection matrix (3.22); after this identification, one obtains the same fermionic correlator, with four spin fields, of the two-charge case [36] and so the modified reflection matrix brings all the new terms with respect to the two-charge calculation.

Since the open string condensate $\bar{\mu}^{(A}\mu^{B)}$ under consideration is invariant under the $SO(4)$ Lorentz group of the T^4 torus, we can restrict ourselves to $SO(4)$ invariant components of $\mathcal{F}_{\hat{A}\hat{B}}$, which, by using the conventions of Appendix C, are $\mathcal{F}_{AB[\dot{\alpha}\dot{\beta}]}$, $\mathcal{F}^{AB[\alpha\beta]}$. In addition the RR components $\mathcal{F}^{AB[\alpha\beta]}$ can be discarded by noticing that the only $SO(6)$ singlet $\epsilon_{ABCD}\mu^A\bar{\mu}^B\mathcal{F}^{CD[\alpha\beta]}$ vanishes for the symmetric open string condensate $\bar{\mu}^{(A}\mu^{B)}$ we consider. As a result, the open string condensate under analysis contributes only to the emission of

$$W_R^{\text{ef}} = \frac{1}{8} (\mathcal{F}\mathcal{R})_{AB}\epsilon_{\dot{\alpha}\dot{\beta}} e^{-\frac{\varphi}{2}} S^A S^{\dot{\alpha}}(z) e^{-\frac{\varphi}{2}} S^B S^{\dot{\beta}}(\bar{z}) , \quad (4.22)$$

with $\mathcal{R} = \mathcal{R}_{\text{D1}}$ (or alternatively $\mathcal{R} = \mathcal{R}_{\text{D5}}$) and where

$$\begin{aligned} \mathcal{F}_{AB} &= \frac{1}{24!} F_{Iabcd}^{(5)} (C_{10} \Gamma_{(10)}^{Iabcd})_{AB\dot{\alpha}} + \sum_{n=1,3,5} \frac{1}{2n!} F_{I_1..I_n}^{(n)} (C_{10} \Gamma_{(10)}^{I_1..I_n})_{AB\dot{\alpha}} \\ &= F_{I_{5678}}^{(5)} (C \Gamma^I)_{AB} + \sum_{n=1,3,5} \frac{1}{n!} F_{I_1..I_n}^{(n)} (C \Gamma^{I_1..I_n})_{AB} . \end{aligned} \quad (4.23)$$

In the last line above we have used the facts that $\Gamma_{(10)}^{5678} = -\Gamma_{(10)}^{\bar{3}\bar{4}\bar{4}\bar{1}} = 1_{(6)} \otimes (-\gamma^{ND})$ and that $-\gamma^{ND}$ is 1 on the indices $\dot{\alpha}$ (see (C.3)). Lorentz invariance again fixes the form of \mathcal{A}_R up to a constant calculated in [36], giving

$$\mathcal{A}_R^{\text{D1-D5}} = \frac{i\pi}{2} V_u \int_0^{L_T} d\hat{v} e^{-ik \cdot f(\hat{v})} \bar{\mu}^A (C^{-1} \mathcal{F}\mathcal{R}_0 C^{-1})_{AB} \mu^B , \quad (4.24)$$

where \mathcal{R}_0 is the $SO(1,5)$ part of the spinorial reflection matrix \mathcal{R} given in (3.22), i.e.

$$\mathcal{R}_0 = \frac{1}{2} \Gamma^{uv} + \dot{f}^i(v) \Gamma^{iv} . \quad (4.25)$$

The couplings coming from the $\frac{1}{2} \Gamma^{uv}$ term are the two-charge D1-D5 contributions; these were calculated in [36] and contribute first at order $1/r^3$. The terms coming from the $\dot{f}^i(v) \Gamma^{iv}$ term are new D1-D5-P terms and contribute first at order $1/r^4$.

Expanding the amplitude in terms of the R-R gauge potentials using (4.23) and (4.21) we find the following coupling to the canonically normalized R-R potentials:

$$\mathcal{A}_R^{\text{D1-D5}} = 4\pi V_u \int_0^{L_T} d\hat{v} e^{-ik \cdot f(\hat{v})} k^l \left[(\hat{C}^{(2)})^{vj} v_{vlj} - (\hat{C}^{(2)})^{uj} v_{ulj} \right]$$

$$\begin{aligned}
& -2\hat{C}^{(0)} v_{ulj} \dot{f}^j(\hat{v}) + (\hat{C}^{(2)})^{uv} v_{ulj} \dot{f}^j(\hat{v}) - 2(\hat{C}^{(2)})^{ij} \left(v_{uli} \dot{f}^j(\hat{v}) - v_{ulj} \dot{f}^i(\hat{v}) \right) \\
& - 2(\hat{C}^{(4)})^{5678} v_{ulj} \dot{f}^j(\hat{v}) + (\hat{C}^{(4)})^{uvij} \left(v_{uli} \dot{f}^j(\hat{v}) - v_{ulj} \dot{f}^i(\hat{v}) \right) \Big] \quad (4.26)
\end{aligned}$$

where the first line is the two-charge D1-D5 contribution, and the other terms are the new D1-D5-P contributions, which vanish in each of the two-charge limits.

4.3 Geometry from string amplitudes

We now use the amplitudes calculated in the previous section to read off the couplings with the canonically normalized fields and then to derive the geometric backreaction of the microstate at large distances. As usual the couplings between the D-brane configuration and the perturbative states is given simply by the first variation of the string amplitude

$$\hat{h}_{\mu\nu} = \frac{1}{2} \frac{\delta \mathcal{A}_{\text{NS}}}{\delta \hat{h}^{\mu\nu}} \quad (\mu < \nu), \quad \hat{h}_{\mu\mu} = \frac{\delta \mathcal{A}_{\text{NS}}}{\delta \hat{h}^{\mu\mu}} \quad (\text{no sum over } \mu), \quad (4.27)$$

$$\hat{b}_{\mu\nu} = \frac{\delta \mathcal{A}_{\text{NS}}}{\delta \hat{b}^{\mu\nu}} \quad (\mu < \nu), \quad \hat{C}_{\mu_1 \dots \mu_n}^{(n)} = \frac{\delta \mathcal{A}_{\text{R}}}{\delta \hat{C}^{(n)}_{\mu_1 \dots \mu_n}} \quad (\mu_1 < \mu_2 < \dots < \mu_n), \quad (4.28)$$

where all the fields are set to zero after the variations. The NS-NS amplitude is the combination of the results in (4.7), (4.9) and (4.15), plus higher order corrections

$$\mathcal{A}_{\text{NS}} = \frac{1}{V_4 V_u 2\pi R} \left(\mathcal{A}_{\text{NS}}^{\text{D1}} + \mathcal{A}_{\text{NS}}^{\text{D5}} + \mathcal{A}_{\text{NS}}^{\text{D1-D5}} + \dots \right), \quad (4.29)$$

where the prefactor is needed to cancel the volume of the directions where the momentum of the emitted closed string is set to zero. A similar equation holds in the R-R sector where one must combine (4.10), (4.11) and (4.26).

If we indicate with $a_{\mu_1 \dots \mu_n}(k)$ a generic coupling appearing in (4.27) and (4.28), then we can derive the geometric backreaction of the configuration by sewing a standard propagator $-i/k^2$ and then taking the Fourier transform to rewrite the result in configuration space,

$$a_{\mu_1 \dots \mu_n}(x) = \int \frac{d^4 k}{(2\pi)^4} \left(-\frac{i}{k^2} \right) a_{\mu_1 \dots \mu_n}(k) e^{ik \cdot x}. \quad (4.30)$$

A common feature of the string couplings derived in the previous section is the presence of a f -dependent exponential factor ($e^{-ik \cdot f}$), which combines with the $e^{ik \cdot x}$ Fourier transformation. Thus the contributions of Section 4.1 and of 4.2 will involve the following integrals (4.31) and (4.32) respectively,

$$\int \frac{d^4 k}{(2\pi)^4} \left(-\frac{i}{k^2} \right) e^{ik^i(x^i - f^i)} = \frac{-i}{4\pi^2} \frac{1}{|x^i - f^i|^2}, \quad (4.31)$$

$$\int \frac{d^4 k}{(2\pi)^4} \left(-\frac{i}{k^2} \right) k^l e^{ik^i(x^i - f^i)} = \frac{1}{2\pi^2} \frac{x^l - f^l}{|x^i - f^i|^4} \quad (4.32)$$

where in the above we have used the abuse of notation

$$|x^i - f^i|^2 = \sum_{i=1}^4 (x^i - f^i)^2. \quad (4.33)$$

We now wish to compare the string result with the ansatz of Section 2, so we must change from the canonically normalized (hatted) fields which have propagators $1/k^2$ to the fields appearing in the supergravity action

$$g = \eta + 2\kappa\hat{h}, \quad B = \sqrt{2}\kappa\hat{b}, \quad \phi = \sqrt{2}\kappa\hat{\phi}, \quad C^{(n)} = \sqrt{2}\kappa\hat{C}^{(n)}. \quad (4.34)$$

We then use Eqs. (4.7), (4.9) and (4.17) in Eq. (4.30) to derive the metric induced by our D-brane configuration. We thus obtain the following dilaton and metric components (in the Einstein frame):

$$e^\phi = 1 + \frac{1}{L_T} \int_0^{L_T} \frac{(Q_1 - Q_5)}{2|x^i - f^i|^2} d\hat{v}, \quad (4.35)$$

$$g_{uj} = \frac{1}{L_T} \int_0^{L_T} \mathbf{v}_{ulj} \frac{x^l - f^l}{|x^i - f^i|^4} d\hat{v}, \quad (4.36)$$

$$g_{vj} = \frac{1}{L_T} \int_0^{L_T} \left[\frac{x^l - f^l}{|x^i - f^i|^4} \left(\mathbf{v}_{vlj} + 2|\dot{f}^i|^2 \mathbf{v}_{ulj} \right) - \frac{(Q_1 + Q_5)\dot{f}_j}{|x^i - f^i|^2} \right] d\hat{v}, \quad (4.37)$$

$$g_{ij} = \delta_{ij} \left(1 + \frac{1}{L_T} \int_0^{L_T} \frac{(Q_1 + 3Q_5)}{4|x^i - f^i|^2} d\hat{v} \right) - \frac{2}{L_T} \int_0^{L_T} \left(\frac{\dot{f}_j(x^l - f^l)}{|x^i - f^i|^4} \mathbf{v}_{uli} + \frac{\dot{f}_i(x^l - f^l)}{|x^i - f^i|^4} \mathbf{v}_{ulj} \right) d\hat{v} \quad (4.38)$$

$$g_{ab} = \delta_{ab} \left(1 + \frac{1}{L_T} \int_0^{L_T} \frac{(Q_1 - Q_5)}{4|x^i - f^i|^2} d\hat{v} \right) \quad (4.39)$$

$$g_{uv} = -\frac{1}{2} + \frac{1}{L_T} \int_0^{L_T} \frac{(3Q_1 + Q_5)}{8|x^i - f^i|^2} d\hat{v} - \frac{1}{L_T} \int_0^{L_T} \mathbf{v}_{ulj} \frac{\dot{f}^j(x^l - f^l)}{|x^i - f^i|^4} d\hat{v}, \quad (4.40)$$

$$g_{vv} = \frac{1}{L_T} \int_0^{L_T} \left[-2\mathbf{v}_{vlj} \frac{\dot{f}^j(x^l - f^l)}{|x^i - f^i|^4} + \frac{(Q_1 + Q_5)|\dot{f}^j|^2}{|x^i - f^i|^2} \right] d\hat{v} \quad (4.41)$$

where in order to make the equations more readable, we have absorbed some factors in the $\bar{\mu}\mu$ condensate

$$\mathbf{v}_{IJK} = -\frac{2\sqrt{2}n_w\kappa}{\pi V_4} v_{IJK} \quad (4.42)$$

and we have introduced the standard combinations Q_1 and Q_5 ,

$$Q_1 = \frac{n_w \tau_1 \kappa^2}{2\pi^2 V_4} = \frac{(2\pi)^4 g_s \alpha'^3 n_w}{V_4}, \quad Q_5 = \frac{n_w \tau_5 \kappa^2}{2\pi^2} = \alpha' g_s n_w. \quad (4.43)$$

In the above, if we set to zero the condensate v_{IJK} for the open string stretched between the D1 and the D5 branes we recover the a linear combination of two-charge solutions D1/P and D5/P, as already verified in [41]. Another limit consists of switching off the momentum charge by setting $f = 0$. In this case we recover the solutions appropriate for the two-charge microstates D1/D5. Of particular interest are the new contributions that vanish in both limits.

In order to obtain the large distance behaviour of the supergravity fields listed above, it is sufficient to expand the denominators for $x^i \gg f^i$. Some of the leading contributions vanish because of the periodicity of the profile, which implies $\int_0^{L_T} f^i(\hat{v}) d\hat{v} = 0$ meaning that the first non-trivial corrections are proportional to the moment of the wave $\int_0^{L_T} \dot{f}^i f^j d\hat{v}$. As discussed in some details at the beginning of this section, the $1/r^4$ behaviour obtained in this way should be universal for all microstates which have the same momentum profile on the D1 and the D5-branes. Here we decided to keep the exact dependence on f^i , as it follows from the string computation, since these formulae are relevant for a smaller class of microstate where the condensate v_{IJK} is small (or in other words, states which are localized near the origin of the classical Higgs branch). This information will provide a useful guide when generalizing the perturbative $1/r$ expansion to a full non-linear supergravity solution. For this reason, in the following, we will keep the exact dependence of the string results on the momentum profile.

In a similar way, from Eq. (4.18) we obtain the non-zero components of the B-field:

$$B_{vj} = \frac{2}{L_T} \int_0^{L_T} |\dot{f}|^2 \mathbf{v}_{ujl} \frac{x^l - f^l}{|x^i - f^i|^4} d\hat{v}, \quad (4.44)$$

$$B_{ij} = -\frac{2}{L_T} \int_0^{L_T} \left(\mathbf{v}_{uli} \frac{\dot{f}_j(x^l - f^l)}{|x^i - f^i|^4} - \mathbf{v}_{ulj} \frac{\dot{f}_i(x^l - f^l)}{|x^i - f^i|^4} \right) d\hat{v}, \quad (4.45)$$

$$B_{uv} = -\frac{1}{L_T} \int_0^{L_T} \mathbf{v}_{ulj} \frac{\dot{f}^j(x^l - f^l)}{|x^i - f^i|^4} d\hat{v}. \quad (4.46)$$

Notice that all these contributions are new in the sense that they disappear in the two-charge limits where either f^i or v_{IJK} are set to zero. On the contrary we have seen that the dilaton does not receive such new contributions from the mixed disk diagram in Figure 1.

Following the same approach with the R-R fields, we use Eqs. (4.10), (4.11) and (4.26)

in Eq. (4.30) to derive the backreaction of the microstate under analysis in this sector. We find the nonzero R-R fields:

$$C^{(0)} = \frac{2}{L_T} \int_0^{L_T} \mathbf{v}_{ulj} \frac{\dot{f}^j(x^l - f^l)}{|x^i - f^i|^4} d\hat{v}, \quad (4.47)$$

$$C_{vj}^{(2)} = \frac{1}{L_T} \int_0^{L_T} \left[-\mathbf{v}_{ulj} \frac{x^l - f^l}{|x^i - f^i|^4} + Q_1 \frac{\dot{f}_j}{|x^i - f^i|^2} \right] d\hat{v}, \quad (4.48)$$

$$C_{uj}^{(2)} = \frac{1}{L_T} \int_0^{L_T} \mathbf{v}_{ulj} \frac{x^l - f^l}{|x^i - f^i|^4} d\hat{v}, \quad (4.49)$$

$$C_{uv}^{(2)} = \frac{1}{L_T} \int_0^{L_T} \left[-\mathbf{v}_{ulj} \frac{\dot{f}^j(x^l - f^l)}{|x^i - f^i|^4} + \frac{Q_1}{2|x^i - f^i|^2} \right] d\hat{v}, \quad (4.50)$$

$$C_{ij}^{(2)} = \frac{2}{L_T} \int_0^{L_T} \left[\mathbf{v}_{uli} \frac{\dot{f}_j(x^l - f^l)}{|x^i - f^i|^4} - \mathbf{v}_{ulj} \frac{\dot{f}_i(x^l - f^l)}{|x^i - f^i|^4} \right] d\hat{v}, \quad (4.51)$$

$$C_{uvij}^{(4)} = -\frac{1}{L_T} \int_0^{L_T} \left[\mathbf{v}_{uli} \frac{\dot{f}_j(x^l - f^l)}{|x^i - f^i|^4} - \mathbf{v}_{ulj} \frac{\dot{f}_i(x^l - f^l)}{|x^i - f^i|^4} \right] d\hat{v}, \quad (4.52)$$

$$C_{5678}^{(4)} = \frac{2}{L_T} \int_0^{L_T} \mathbf{v}_{ulj} \frac{\dot{f}^j(x^l - f^l)}{|x^i - f^i|^4} d\hat{v}, \quad (4.53)$$

$$C_{vj5678}^{(6)} = \frac{Q_5}{L_T} \int_0^{L_T} \frac{\dot{f}_j}{|x^i - f^i|^2} d\hat{v}, \quad (4.54)$$

$$C_{uv5678}^{(6)} = \frac{1}{L_T} \int_0^{L_T} \frac{Q_5}{2|x^i - f^i|^2} d\hat{v}. \quad (4.55)$$

In the next section we read off from these fields the string contribution to the fields parameterizing the supergravity ansatz of Section 2.

5 Comparison to supergravity

We will now verify that the fields derived from the string amplitudes satisfy the supergravity constraints obtained in Section 2.

The supergravity analysis was performed in the large r limit, keeping only terms up to order $1/r^4$. One should thus apply the supergravity equations to the large

r expansion of the string results of the previous section. It turns out, however, that one can keep the full r dependence of the string results and still satisfy⁴ the approximate supergravity equations of Section 2. This is what we will show in the following. We remind the reader that full r dependence of the supergravity fields is meaningful in describing the small $g_s N$ and small \mathbf{v}_{IJK} limit, i.e. the weak gravity regime and the region of the Higgs branch infinitesimally close to its intersection with the Coulomb branch. If one is interested in the full black hole regime (large $g_s N$ and finite \mathbf{v}_{IJK}), one should keep only the large r limit (up to $1/r^4$ order) of the results we present below.

In order to make equations more compact, it is useful to define the the following integrals

$$\mathcal{I} = \frac{1}{L_T} \int_0^{L_T} d\hat{v} \frac{1}{|x^i - f^i|^2}, \quad \tilde{\mathcal{I}} = \frac{1}{L_T} \int_0^{L_T} d\hat{v} \frac{|\dot{f}^j|^2}{|x^i - f^i|^2}, \quad \mathcal{I}_j = \frac{1}{L_T} \int_0^{L_T} d\hat{v} \frac{\dot{f}_j}{|x^i - f^i|^2}. \quad (5.1)$$

The properties

$$\partial_i^2 \mathcal{I} = \partial_i^2 \tilde{\mathcal{I}} = \partial_i^2 \mathcal{I}_j = 0, \quad \partial_i \mathcal{I}_i = 0 \quad (5.2)$$

easily follow from the definitions above and the fact that $f_i(\hat{v})$ is a periodic function.

We first extract from the string results of Section 4 the metric functions used to parameterize the general supergravity ansatz of Section 2 and then verify that they obey the constraints from supersymmetry and the equations of motion. In doing this we will only keep terms up to first order in the condensate \mathbf{v}_{IJK} and in the Q_1 and Q_5 charges.

As the supergravity ansatz is given in the string frame, it is useful to translate the string results for the metric, given in Einstein frame, into string frame. At our order of approximation, if we denote by $\eta_{\mu\nu} + h_{\mu\nu}$ and by $g_{\mu\nu}$ the string frame and Einstein frame metrics, one has

$$\eta_{\mu\nu} + h_{\mu\nu} = g_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} \phi. \quad (5.3)$$

From eqs. (4.35)-(4.41), one then finds that the world-sheet prediction for the metric

⁴This happens because the approximate constraints of Section 2 are valid up to terms of second order in the condensate \mathbf{v}_{IJK} and thus in the same approximation in which the string results have been derived.

in string frame is

$$h_{uj} = -\frac{1}{2} \mathbf{v}_{ulj} \partial_l \mathcal{I}, \quad (5.4)$$

$$h_{vj} = -\frac{1}{2} \mathbf{v}_{vlj} \partial_l \mathcal{I} - \mathbf{v}_{ulj} \partial_l \tilde{\mathcal{I}} - (Q_1 + Q_5) \mathcal{I}_j, \quad (5.5)$$

$$h_{ij} = \frac{1}{2} (Q_1 + Q_5) \mathcal{I} \delta_{ij} + \mathbf{v}_{uli} \partial_l \mathcal{I}_j + \mathbf{v}_{ulj} \partial_l \mathcal{I}_i, \quad (5.6)$$

$$h_{ab} = \frac{1}{2} (Q_1 - Q_5) \mathcal{I} \delta_{ab}, \quad (5.7)$$

$$h_{uv} = \frac{1}{4} (Q_1 + Q_5) \mathcal{I} + \frac{1}{2} \mathbf{v}_{ulk} \partial_l \mathcal{I}_k, \quad (5.8)$$

$$h_{vv} = \mathbf{v}_{vlk} \partial_l \mathcal{I}_k + (Q_1 + Q_5) \tilde{\mathcal{I}}. \quad (5.9)$$

We should also dualize the RR 6-form computed on the string side into a 2-form:

$$\begin{aligned} C^{(6)} &= Q_5 \left(\frac{1}{2} \mathcal{I} du \wedge dv + \mathcal{I}_i dv \wedge dx_i \right) \wedge dz^4 \Rightarrow \\ d\hat{C}^{(2)} &= - * dC^{(6)} = Q_5 \epsilon_{ijkl} (\partial_l \mathcal{I} dx^k + \partial_k \mathcal{I}_l dv) \wedge dx^i \wedge dx^j. \end{aligned} \quad (5.10)$$

One can check that, thanks to (5.2), the forms

$$\epsilon_{ijkl} \partial_l \mathcal{I} dx^i \wedge dx^j \wedge dx^k \quad \text{and} \quad \epsilon_{ijkl} \partial_k \mathcal{I}_l dx^i \wedge dx^j \quad (5.11)$$

are d -closed, and hence one can define a 2-form $\frac{1}{2} \hat{\mathcal{I}}_{ij} dx^i \wedge dx^j$ and a 1-form $\hat{\mathcal{I}}_i dx^i$ such that

$$d\left(\frac{1}{2} \hat{\mathcal{I}}_{ij} dx^i \wedge dx^j\right) = \epsilon_{ijkl} \partial_l \mathcal{I} dx^i \wedge dx^j \wedge dx^k, \quad d(\hat{\mathcal{I}}_i dx^i) = \epsilon_{ijkl} \partial_k \mathcal{I}_l dx^i \wedge dx^j. \quad (5.12)$$

Then the dual of $C^{(6)}$ is

$$\hat{C}^{(2)} = Q_5 \left(\frac{1}{2} \hat{\mathcal{I}}_{ij} dx^i \wedge dx^j - \hat{\mathcal{I}}_i dv \wedge dx^i \right). \quad (5.13)$$

This gives additional contributions to the $C_{vj}^{(2)}$ and $C_{ij}^{(2)}$ of eqs. (4.48), (4.51), so that in total one has

$$C_{vj}^{(2)} = \frac{1}{2} \mathbf{v}_{vlj} \partial_l \mathcal{I} + Q_1 \mathcal{I}_j - Q_5 \hat{\mathcal{I}}_j, \quad (5.14)$$

$$C_{ij}^{(2)} = -\mathbf{v}_{uli} \partial_l \mathcal{I}_j - \mathbf{v}_{ulj} \partial_l \mathcal{I}_i + Q_5 \hat{\mathcal{I}}_{ij}. \quad (5.15)$$

We can now compute the various metric coefficients that appear in the supergravity

ansatz (2.1), (2.3), (2.6) :

$$Z_1 = 1 + 2 h_{uv} + h_{aa} = 1 + Q_1 \mathcal{I} + \mathbf{v}_{ulk} \partial_l \mathcal{I}_k, \quad (5.16)$$

$$Z_2 = 1 + 2 h_{uv} - h_{aa} = 1 + Q_5 \mathcal{I} + \mathbf{v}_{ulk} \partial_l \mathcal{I}_k, \quad (5.17)$$

$$Z_3 = 1 + h_{vv} = 1 + (Q_1 + Q_5) \tilde{\mathcal{I}} + \mathbf{v}_{ulk} \partial_l \mathcal{I}_k, \quad (5.18)$$

$$a_3 = -2 h_{ui} dx^i = \mathbf{v}_{uli} \partial_l \mathcal{I} dx^i, \quad (5.19)$$

$$k = -(h_{ui} + h_{vi}) dx^i = \left[(Q_1 + Q_5) \mathcal{I}_i + \frac{1}{2} (\mathbf{v}_{uli} + \mathbf{v}_{vli}) \partial_l \mathcal{I} + \mathbf{v}_{uli} \partial_l \tilde{\mathcal{I}} \right] dx^i, \quad (5.20)$$

$$\begin{aligned} ds_4^2 &= (\delta_{ij} + h_{ij} - 2 h_{vu} \delta_{ij}) dx^i dx^j \\ &= [\delta_{ij} + \mathbf{v}_{uli} \partial_l \mathcal{I}_j + \mathbf{v}_{ulj} \partial_l \mathcal{I}_i - \delta_{ij} \mathbf{v}_{ulk} \partial_l \mathcal{I}_k] dx^i dx^j, \end{aligned} \quad (5.21)$$

$$D = e^{2\phi} = 1 + (Q_1 - Q_5) \mathcal{I}, \quad (5.22)$$

$$b_0 = -2 B_{uv} = -\mathbf{v}_{ulk} \partial_l \mathcal{I}_k, \quad (5.23)$$

$$b_1 = (B_{ui} - B_{vi}) dx^i = \mathbf{v}_{uil} \partial_l \tilde{\mathcal{I}} dx^i, \quad (5.24)$$

$$\tilde{b}_1 = -(B_{ui} + B_{vi}) dx^i = \mathbf{v}_{uil} \partial_l \tilde{\mathcal{I}} dx^i, \quad (5.25)$$

$$b_2 = \frac{1}{2} B_{ij} dx^i \wedge dx^j = \mathbf{v}_{uli} \partial_l \mathcal{I}_j dx^i \wedge dx^j, \quad (5.26)$$

$$c = C^{(0)} = -\mathbf{v}_{ulk} \partial_l \mathcal{I}_k, \quad (5.27)$$

$$\tilde{Z}_1 = 1 + 2 C_{uv}^{(2)} = 1 + Q_1 \mathcal{I} + \mathbf{v}_{ulk} \partial_l \mathcal{I}_k, \quad (5.28)$$

$$a_1 = (-h_{ui} - h_{vi} + C_{ui}^{(2)} - C_{vi}^{(2)}) dx^i = Q_5 (\mathcal{I}_i + \hat{\mathcal{I}}_i) dx^i + \mathbf{v}_{uli} \partial_l \tilde{\mathcal{I}} dx^i, \quad (5.29)$$

$$\tilde{a}_1 = (h_{ui} - h_{vi} - C_{ui}^{(2)} - C_{vi}^{(2)}) dx^i = Q_5 (\mathcal{I}_i + \hat{\mathcal{I}}_i) dx^i + \mathbf{v}_{uli} \partial_l \tilde{\mathcal{I}} dx^i, \quad (5.30)$$

$$\tilde{\gamma}_2 = \frac{1}{2} C_{ij}^{(2)} dx^i \wedge dx^j = \frac{1}{2} Q_5 \hat{\mathcal{I}}_{ij} dx^i \wedge dx^j - \mathbf{v}_{uli} \partial_l \mathcal{I}_j dx^i \wedge dx^j, \quad (5.31)$$

$$f = C_{5678}^{(4)} = -\mathbf{v}_{ulk} \partial_l \mathcal{I}_k. \quad (5.32)$$

Finally we need to derive the 0-forms \tilde{b}_0 and \tilde{Z}_2 defined in (2.5). One has

$$\begin{aligned} d\tilde{b}_0 &= - *_4 db_2 = - *_4 (\mathbf{v}_{uli} \partial_k \partial_l \mathcal{I}_j dx^i \wedge dx^j \wedge dx^k) = -\epsilon_{mijk} \mathbf{v}_{uli} \partial_k \partial_l \mathcal{I}_j dx^m \\ &= \frac{1}{2} \epsilon_{mijk} \epsilon_{lipq} \mathbf{v}_{upq} \partial_k \partial_l \mathcal{I}_j dx^m \\ &= \mathbf{v}_{ujk} \partial_k \partial_l \mathcal{I}_j dx^l + \mathbf{v}_{ukl} \partial_k \partial_j \mathcal{I}_j dx^l + \mathbf{v}_{ulj} \partial_k \partial_k \mathcal{I}_j dx^l \\ &= -d(\mathbf{v}_{ulk} \partial_l \mathcal{I}_k) \Rightarrow \tilde{b}_0 = -\mathbf{v}_{ulk} \partial_l \mathcal{I}_k, \end{aligned} \quad (5.33)$$

where we have used the anti-self-duality of \mathbf{v}_{uij} and the properties (5.2) of \mathcal{I}_i . Similarly one finds that

$$d\tilde{Z}_2 = - *_4 d\tilde{\gamma}_2 \Rightarrow \tilde{Z}_2 = 1 + Q_5 \mathcal{I} + \mathbf{v}_{ulk} \partial_l \mathcal{I}_k, \quad (5.34)$$

where we have picked the solution for \tilde{Z}_2 that goes to 1 at infinity, and have used, to derive the Q_5 -proportional term, the definition (5.12) of $\hat{\mathcal{I}}_{ij}$.

We now have all the ingredients to verify the supergravity constraints. The equalities $\tilde{Z}_1 = Z_1$, $\tilde{Z}_2 = Z_2$, $D = \frac{Z_1}{Z_2}$, $b_0 = \tilde{b}_0 = c = f$, $\tilde{a}_1 = a_1$, $\tilde{b}_1 = b_1$ are evidently satisfied

by the metric coefficients listed above. The self-duality condition $da_3 = *_4 da_3$ follows from anti-self-duality of \mathbf{v}_{uij} and the fact that \mathcal{I} is harmonic (5.2):

$$\begin{aligned}
*_4 da_3 &= *_4(\mathbf{v}_{uli} \partial_j \partial_l \mathcal{I} dx^j \wedge dx^i) = \frac{1}{2} \epsilon_{kmji} \mathbf{v}_{uli} \partial_j \partial_l \mathcal{I} dx^k \wedge dx^m \\
&= -\frac{1}{4} \epsilon_{kmji} \epsilon_{lipq} \mathbf{v}_{upq} \partial_j \partial_l \mathcal{I} dx^k \wedge dx^m \\
&= -\frac{1}{2} (2 \mathbf{v}_{umj} \partial_j \partial_k \mathcal{I} dx^k \wedge dx^m + \mathbf{v}_{ukm} \partial_j \partial_j \mathcal{I} dx^k \wedge dx^m) \\
&= da_3.
\end{aligned} \tag{5.35}$$

The proof of the self-duality of db_1 is identical, with the replacement $\mathcal{I} \rightarrow \tilde{\mathcal{I}}$. The same identity, plus the definition of $\tilde{\mathcal{I}}_i$ (5.12), shows that da_1 is self-dual. The fact that Z_1, Z_2, Z_3, b_0 and k are harmonic follows from the harmonicity of $\mathcal{I}, \tilde{\mathcal{I}}$, and \mathcal{I}_i (5.2). We are left to show that the 4D metric ds_4^2 given in (5.21) is hyper-Kahler: this also follows from anti-self-duality of \mathbf{v}_{uij} and the properties of \mathcal{I}_i , and we give the details of the proof in Appendix D.

The supergravity ansatz of Section 2 reduces to the class of supergravity solutions that have been used in the literature [13] to describe black hole microstates when $b_0 = \tilde{b}_0 = c = f = 0$ and $b_1 = \tilde{b}_1 = 0$. We have seen that the string amplitude computation predicts that the class of D-brane configurations with equal D1 and D5 profiles emits non-zero values of these latter fields, and thus cannot be described by the existing microstate geometries. The fields b_1 and b_0 (that first appear at order $1/r^3$ and $1/r^4$, respectively) represent new types of dipole and quadrupole moments, proportional to both the D1-D5 vev \mathbf{v}_{uij} and the derivative of the string profile \dot{f}_i , and thus vanish when any one of the three charges vanishes. This is in contrast with the three types of dipole moments of the existing microstate solutions, each of which survives in one of the three 2-charge limits. Since the new multipole moments involve all three charges, it is difficult to use dualities to relate them to a simpler system, as can be done for the moments involving only two charges at a time. Another interesting outcome of our calculation is that it predicts that the 4D base metric ds_4^2 , which is simply the flat metric on \mathbb{R}^4 in the 2-charge case, is a *non-trivial* hyper-Kahler metric when all three charges are non-vanishing. The non-flatness of the base metric for 3-charge microstate geometries was already noted in the particular solution of [31], and it had remained until now a largely unexplained phenomenon. It is nice to see that our approach neatly predicts this feature.

6 Discussion

In this paper we showed how to extract information on the geometrical backreaction of D-brane bound states, in the regime of finite gravitational coupling, from

perturbative string amplitudes. The string amplitudes of interest involve both open and closed strings; the open strings determine the state of the D-brane configuration and the closed strings specify the supergravity field under consideration. The most interesting contributions come from disk amplitudes that mix different types of boundary conditions, in a spirit very similar to the stringy description of classical gauge instantons of [56]. In this setup, the open strings stretched between the instantonic and the physical branes are part of the instanton moduli and so the physical observables are obtained after integrating over these fields. For instance, recently [57] considered a $\mathcal{N} = 2$ superconformal setup and derived the backreaction on the axion-dilaton field due to the presence of D(−1)-branes. As seen in the two-charge cases [36, 41], in our construction the open string vevs contain the data specifying the microstate and no integration over the open string fields is necessary. We saw that also the gravitational couplings of three-charge microstates are determined by the open string data, which in our case are encoded by the functions $f^i(\hat{v})$ and the condensate v_{IJK} . Once these couplings are derived from string theory, the leading gravitational backreaction is obtained by solving the free bulk equations of motion. As a consistency check, we also showed explicitly that the bulk configurations derived in this way are consistent with the type IIB equations of motion and preserve four supersymmetries, at least up to fourth order in the $1/r$ expansion.

The results presented in this paper focus on a particular class of three-charge bound states which has a simple world-sheet description, as described in Section 3.4. Most likely a typical microstate of the D1-D5-P system will not be in this class of configuration. In addition, our result about the large distance behaviour of the supergravity fields are non-trivial only if the wave profile is slowly varying and its moments, such as $\int_0^{L_T} f^i f^j d\hat{v}$, are sizable. Again this is certainly not the case for a generic microstate, where the direction (in the \mathbb{R}^4) of the modes of the profile will be randomly distributed. As usual we hope to learn something about the backreaction of a typical microstate, even if we start by focusing on an atypical case described by semiclassical data such as the profile functions f^i .

It is interesting to notice that the simplicity at the microscopic level is not reflected in a particularly compact supergravity solution. On the contrary, the geometric backreaction for a D1 and D5-brane bound state with equal oscillations contains new types of multipole moments that do not appear in the class of 1/8-BPS solutions studied in [20]. In discussing the regime of validity of our perturbative string calculation, we have given our reasons for believing that this more general type of asymptotic behavior applies also to states deep in the Higgs branch and for large $g_s N$.

It would be of course interesting to see whether we can engineer a D-brane configuration which emits *only* the fields excited in the ansatz of [20]. In our case, this

would require to switch off all fields in Section 4 proportional to $\mathbf{v}_{ujl} \dot{f}^j \partial_l (1/r^2)$. For instance this would happen if the profile function f^i were made of two disconnected circles in the (x_1, x_2) and (x_3, x_4) planes, but this is not an allowed configuration for a microstate. It does not seem to be simple to satisfy this requirement with an allowed profile. This also means that the microstate solutions [30] are not described by our D-brane configurations.

By extrapolating our solution to the case where the profile functions on the D1 and D5 branes are different, one may write an educated guess for the structure of a configuration where f^{D1} is a circle in the (x_1, x_2) plane and f^{D5} is given by the same circle but now in the (x_3, x_4) plane. It is possible that this is the D-brane configuration whose backreaction reduces to the ansatz in [20]. However, in order to analyze explicitly this case, we need first to derive the possible states for an open string stretched between D1 and D5 branes with different profiles.

Another interesting future line of development is to keep focusing on the class of configurations analyzed in this paper and derive a full non-linear ansatz solving the type IIB supersymmetry variations and equations of motion. The final goal would be to extend the relations (5.16)-(5.32) to all orders in the condensate v_{IJK} . The recent proposal of [47], that associates three-charge bound state configurations with functions of two variables, would suggest that the all-order form of the expressions (5.16)-(5.32) could be represented in terms of integrals of the type appearing in Eq. (5.1), but with the profile $f^i(\hat{v})$ replaced by a function of two variables. If this program could be completed, it would represent a major development towards the construction of a family of geometries with enough degrees of freedom to encode for the full entropy of the three-charge black hole.

A more immediate step towards this goal would be to focus on the subclass of configurations with two axial symmetries. Work in progress indicates that exact solutions within this class can be constructed and it would be very interesting to study the simplest explicit solution of this ansatz. This configuration could play the role the solution in [30] played for the ansatz [20]. An analysis of the “near-horizon” limit of such a solution has the potential to provide, via the AdS/CFT correspondence, further evidence that we are really considering the geometrical backreaction of a three-charge microstate.

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A Constraints from supersymmetry

A.1 Killing spinor equations

In our conventions the supersymmetry variations of the gravitino and dilatino in IIB theory, in units where $\kappa = 1$, are

$$\begin{aligned}\delta\psi_M &= \left(\nabla_M - \frac{i}{2}Q_M\right)\epsilon + \frac{i}{192}\Gamma^{M_1\dots M_4}F_{M_1\dots M_4 M}^{(5)}\epsilon \\ &\quad - \frac{1}{96}G_{NPQ}\Gamma_M{}^{NPQ}\epsilon^* + \frac{9}{96}G_{MNP}\Gamma^{NP}\epsilon^*,\end{aligned}\tag{A.1}$$

$$\delta\lambda = i\Gamma^M P_M \epsilon^* + \frac{i}{24}G_{MNP}\Gamma^{MNP}\epsilon,\tag{A.2}$$

where

$$\begin{aligned}P &= \frac{i}{2}e^\phi dC^{(0)} + \frac{1}{2}d\phi, \\ Q &= -\frac{1}{2}e^\phi dC^{(0)}, \\ G &= ie^{\phi/2}(\tau dB - dC^{(2)}), \\ \tau &= C^{(0)} + ie^{-\phi}.\end{aligned}\tag{A.3}$$

We take the supersymmetry parameter ϵ to satisfy the chirality condition

$$\Gamma^{0y12345678}\epsilon = \epsilon,\tag{A.4}$$

where 1, 2, 3, 4 are the directions of \mathbb{R}^4 and 5, 6, 7, 8 the T^4 directions. Correspondingly $F^{(5)}$ satisfies $F^{(5)} = *F^{(5)}$, where the star operation is defined using the orientation $\epsilon_{0y12345678} = 1$. We are using a base in which the 10D gamma matrices are purely imaginary, in which case the conjugate ϵ^* of ϵ is simply given by complex conjugation: $\epsilon^* = \epsilon_1 - i\epsilon_2$, with $\epsilon = \epsilon_1 + i\epsilon_2$ and ϵ_1, ϵ_2 real spinors. We will denote the \mathbb{R}^4 coordinates by $i, j, \dots = 1, 2, 3, 4$ and the T^4 coordinates by $a, b, \dots = 5, 6, 7, 8$.

A.2 Vielbeins, spin connection and gauge fields

To explicitly write the Killing spinor equations one needs the vielbeins and spin connection of the Einstein frame metric ($ds_E^2 = e^{-\phi/2}ds^2$) and the gauge fields for the general ansatz specified in section 2. We give these data below, keeping only the terms that contribute to the large distance expansion up to order $1/r^4$.

The vielbeins are

$$\begin{aligned}e^t &= \frac{1}{(Z_1 Z_2)^{1/4} Z_3^{1/2} D^{1/8}}(dt + k), \quad e^y = \frac{Z_3^{1/2}}{(Z_1 Z_2)^{1/4} D^{1/8}}\left(dy + dt - \frac{dt + k}{Z_3} + a_3\right), \\ e^i &= \frac{(Z_1 Z_2)^{1/4}}{D^{1/8}}\bar{e}^i, \quad e^a = \left(\frac{Z_1}{Z_2}\right)^{1/4} \frac{1}{D^{1/8}}dx^a,\end{aligned}\tag{A.5}$$

with \bar{e}^i the vielbeins of the metric ds_4^2 . The non-trivial components of the spin connection are

$$\begin{aligned}
\omega_{ti} &= \frac{D^{1/8}}{(Z_1 Z_2)^{1/4}} \left(\frac{1}{4} \partial_i \log Z_1 + \frac{1}{4} \partial_i \log Z_2 + \frac{1}{2} \partial_i \log Z_3 + \frac{1}{8} \partial_i \log D \right) e^t \\
&\quad + \frac{1}{2} \frac{D^{1/8}}{(Z_1 Z_2)^{1/4}} \partial_i \log Z_3 e^y - \frac{1}{2} (\partial_i k_j - \partial_j k_i) e^j, \\
\omega_{ty} &= \frac{1}{2} \frac{D^{1/8}}{(Z_1 Z_2)^{1/4}} \partial_i \log Z_3 e^i, \\
\omega_{yi} &= \frac{D^{1/8}}{(Z_1 Z_2)^{1/4}} \left(-\frac{1}{4} \partial_i \log Z_1 - \frac{1}{4} \partial_i \log Z_2 + \frac{1}{2} \partial_i \log Z_3 - \frac{1}{8} \partial_i \log D \right) e^y \\
&\quad + \frac{1}{2} \frac{D^{1/8}}{(Z_1 Z_2)^{1/4}} \partial_i \log Z_3 e^t + \frac{1}{2} (\partial_i a_{3j} - \partial_j a_{3i} - \partial_i k_j + \partial_j k_i) e^j, \\
\omega_{ij} &= \frac{D^{1/8}}{(Z_1 Z_2)^{1/4}} \left(\frac{1}{4} \partial_j \log Z_1 + \frac{1}{4} \partial_j \log Z_2 - \frac{1}{8} \partial_j \log D \right) e^i - (i \leftrightarrow j) \\
&\quad + \frac{1}{2} (\partial_i k_j - \partial_j k_i) e^t - \frac{1}{2} (\partial_i a_{3j} - \partial_j a_{3i} - \partial_i k_j + \partial_j k_i) e^y + \bar{\omega}_{ij}, \\
\omega_{ai} &= \frac{D^{1/8}}{(Z_1 Z_2)^{1/4}} \left(\frac{1}{4} \partial_i \log Z_1 - \frac{1}{4} \partial_i \log Z_2 - \frac{1}{8} \partial_i \log D \right) e^a, \tag{A.6}
\end{aligned}$$

where $\bar{\omega}_{ij}$ is the spin connection of ds_4^2 .

The gauge fields are

$$P = \frac{i}{2} \partial_i c e^i + \frac{1}{4} \frac{D^{1/8}}{(Z_1 Z_2)^{1/4}} \partial_i \log D e^i, \tag{A.7}$$

$$Q = -\frac{1}{2} \partial_i c e^i, \tag{A.8}$$

$$\begin{aligned}
G &= -i \left(i \partial_i b_0 + D^{5/8} (Z_1 Z_2)^{1/4} \frac{\partial_i \tilde{Z}_1}{\tilde{Z}_1^2} \right) e^i \wedge e^t \wedge e^y \\
&\quad - i (-i \partial_i b_{1j} + \partial_i a_{1j} - \partial_i k_j) e^i \wedge e^j \wedge e^y \\
&\quad - i (-i \partial_i \tilde{b}_{1j} + \partial_i \tilde{a}_{1j} + \partial_i a_{3j} - \partial_i k_j) e^i \wedge e^j \wedge e^t \\
&\quad - i \frac{\epsilon_{ijkl}}{3!} \left(-i \partial_l \tilde{b}_0 + \frac{D^{5/8}}{(Z_1 Z_2)^{3/4}} \partial_l \tilde{Z}_2 \right) e^i \wedge e^j \wedge e^k. \tag{A.9}
\end{aligned}$$

We will analyze below the constraints coming from imposing $\delta\psi_M = \delta\lambda = 0$ order by order in the $1/r$ expansion.

A.3 Order $1/r^2$

At order $1/r^2$ the only non-trivial functions are $Z_1 = \tilde{Z}_1, Z_2 = \tilde{Z}_2, Z_3$ and $D = Z_1/Z_2$.⁵ At this order the dilatino equation $\delta\lambda = 0$ becomes

$$i(\partial_i Z_1 - \partial_i Z_2) \Gamma^i \epsilon^* + \partial_i Z_1 \Gamma^{ity} \epsilon + \frac{1}{3!} \epsilon_{ijkl} \partial_l Z_2 \Gamma^{ijk} \epsilon = 0. \quad (\text{A.10})$$

Requiring the coefficients of $\partial_i Z_1$ and $\partial_i Z_2$ to vanish separately gives

$$\Gamma^{ty} \epsilon_1 = -\epsilon_2, \quad \Gamma^{1234} \epsilon_1 = -\epsilon_2. \quad (\text{A.11})$$

No new constraint is imposed by the $M = a$ components of the gravitino equation $\delta\psi_M = 0$. The $M = t$ component of the gravitino equation is

$$\left(\frac{3}{4}\partial_i Z_1 + \frac{1}{4}\partial_i Z_2 + \partial_i Z_3\right) \Gamma^{ti} \epsilon + \partial_i Z_3 \Gamma^{yi} \epsilon - \frac{i}{4} \frac{1}{3!} \epsilon_{jkl} \partial_l Z_2 \Gamma^{tjkl} \epsilon^* + i \frac{3}{4} \partial_i Z_1 \Gamma^{iy} \epsilon^* = 0, \quad (\text{A.12})$$

and one has an equivalent equation from $M = y$. The Z_1 and Z_2 terms vanish thanks to (A.11); the Z_3 term implies:

$$\Gamma^{ty} \epsilon_1 = \epsilon_1, \quad (\text{A.13})$$

which together with (A.11) gives

$$\epsilon_2 = -\epsilon_1. \quad (\text{A.14})$$

Finally the $M = i$ components of the gravitino equation yield

$$\begin{aligned} \partial_i \epsilon + \frac{1}{4} \partial_i Z_3 \Gamma^{ty} \epsilon + \frac{1}{16} \left(\partial_j Z_1 + 3 \partial_j Z_2 \right) \Gamma^{ij} \epsilon + \frac{i}{16} (\partial_j Z_1 \Gamma^{ijty} \epsilon^* - \partial_i Z_2 \Gamma^{1234} \epsilon^*) \\ - i \frac{3}{16} \left(\partial_i Z_1 \Gamma^{ty} \epsilon^* + \frac{1}{2} \epsilon_{ijkl} \partial_j Z_2 \Gamma^{kl} \epsilon^* \right) = 0. \end{aligned} \quad (\text{A.15})$$

Using the constraints derived above, this equation reduces to a differential equation for ϵ_1 , which is solved by

$$\epsilon_1 = Z_1^{-3/16} Z_2^{-1/16} Z_3^{-1/4} \epsilon_0, \quad (\text{A.16})$$

with ϵ_0 a constant spinor.

In summary the spinor satisfies the projection conditions

$$\Gamma^{ty} \epsilon_1 = \epsilon_1, \quad \Gamma^{1234} \epsilon_1 = \epsilon_1, \quad \Gamma^{5678} \epsilon_1 = \epsilon_1, \quad \epsilon_2 = -\epsilon_1, \quad (\text{A.17})$$

where the third constraint follows from the previous ones and the chirality condition. These constraints leave 4 independent components, corresponding to the supersymmetries preserved by a 3-charge black hole. We can use these projection conditions in the computation at order $1/r^3$.

⁵By reversing the sign of the RR fields, one could have also taken $\tilde{Z}_1 = -Z_1$ and $\tilde{Z}_2 = -Z_2$. Sending all the RR fields to minus themselves and $\epsilon \rightarrow \epsilon^*$ leaves the supersymmetry variations (A.1), (A.2) invariant and constitutes a symmetry of the theory. Hence our choice is not restrictive.

A.4 Order $1/r^3$

At order $1/r^3$ the only new constraints coming from supersymmetry are the ones involving the 1-forms. We will analyze these new conditions in the following.

The real and imaginary parts of the dilatino equation, or the $M = a$ components of the gravitino equation, imply, after using (A.17):

$$(\partial_i a_{3j} + \partial_i \tilde{a}_{1j} - \partial_i a_{1j}) \Gamma^{ij} \epsilon_1 = 0, \quad (\text{A.18})$$

and

$$(\partial_i \tilde{b}_{1j} - \partial_i b_{1j}) \Gamma^{ij} \epsilon_1 = 0. \quad (\text{A.19})$$

Since the condition $\Gamma^{1234} \epsilon_1 = \epsilon_1$ implies that

$$\Gamma^{ij} \epsilon_1 = -\frac{1}{2} \epsilon_{ijkl} \Gamma^{kl} \epsilon_1 \quad (\text{A.20})$$

an equation of the form $\omega_{ij} \Gamma^{ij} \epsilon_1 = 0$, for some 2-form ω_{ij} , requires that the anti-self-dual part of ω_{ij} vanish, i.e. that $\omega = *_4 \omega$. Hence the two conditions above are equivalent to

$$(1 - *_4)(da_3 + d\tilde{a}_1 - da_1) = 0, \quad (1 - *_4)(db_1 - d\tilde{b}_1) = 0. \quad (\text{A.21})$$

In an analogous way, the $M = t$ and $M = y$ components of the gravitino equation imply

$$(1 - *_4)(da_1 + 3da_3 + 3d\tilde{a}_1) = 0, \quad (1 - *_4)(db_1 + 3d\tilde{b}_1) = 0, \quad (\text{A.22})$$

and

$$(1 - *_4)(d\tilde{a}_1 - 3da_3 + 3da_1) = 0, \quad (1 - *_4)(d\tilde{b}_1 + 3db_1) = 0. \quad (\text{A.23})$$

Altogether these conditions require that all the 1-forms (apart from k) have self-dual field strengths:

$$(1 - *_4)da_1 = (1 - *_4)d\tilde{a}_1 = (1 - *_4)da_3 = (1 - *_4)db_1 = (1 - *_4)d\tilde{b}_1. \quad (\text{A.24})$$

Let us now consider the $M = i$ components of the gravitino equation: the terms involving the scalars give the same differential equation for ϵ found at order $1/r^2$, and hence ϵ is given by an expression of the form A.16 even at $1/r^3$ order. The terms involving the 1-forms can be simplified by the use of the identity

$$\omega_{jk} \Gamma^{ijk} \epsilon_1 = -2 \omega_{ij} \Gamma^j \epsilon_1, \quad (\text{A.25})$$

valid for any self-dual 2-form ω_{ij} if $\Gamma^{1234} \epsilon_1 = \epsilon_1$. Then the real and imaginary parts of the $M = i$ gravitino equation give

$$(\partial_{[i} \tilde{a}_{1j]} - \partial_{[i} a_{1j]}) \Gamma^j \epsilon_1 = 0, \quad (\text{A.26})$$

and

$$(\partial_{[i} \tilde{b}_{1j]} - \partial_{[i} b_{1j]}) \Gamma^j \epsilon_1 = 0, \quad (\text{A.27})$$

which imply

$$a_1 = \tilde{a}_1, \quad b_1 = \tilde{b}_1. \quad (\text{A.28})$$

A.5 Order $1/r^4$

The equations for the 1-forms are unchanged at order $1/r^4$, therefore we will only discuss the scalar and 4D metric sector below.

The conditions following from the real and imaginary part of the dilatino equation are

$$\partial_i \log D - D^{1/2} \left((Z_1 Z_2)^{1/2} \frac{\partial_i \tilde{Z}_1}{\tilde{Z}_1^2} - (Z_1 Z_2)^{-1/2} \partial_i \tilde{Z}_2 \right) = 0, \quad (\text{A.29})$$

$$2 \partial_i c - \partial_i b_0 - \partial_i \tilde{b}_0 = 0, \quad (\text{A.30})$$

and they imply, using the asymptotic conditions (2.7), that

$$D = \frac{\tilde{Z}_1}{\tilde{Z}_2}, \quad c = \frac{1}{2}(b_0 + \tilde{b}_0). \quad (\text{A.31})$$

The $M = a$ gravitino equation gives

$$2 \partial_i \log \frac{Z_1}{Z_2} - \partial_i \log D - D^{1/2} \left((Z_1 Z_2)^{1/2} \frac{\partial_i \tilde{Z}_1}{\tilde{Z}_1^2} - (Z_1 Z_2)^{-1/2} \partial_i \tilde{Z}_2 \right) = 0, \quad (\text{A.32})$$

$$2 \partial_i f - \partial_i b_0 - \partial_i \tilde{b}_0 = 0. \quad (\text{A.33})$$

Combining these conditions with the previous ones gives the further constraints

$$\frac{\tilde{Z}_1}{\tilde{Z}_2} = \frac{Z_1}{Z_2}, \quad f = c = \frac{1}{2}(b_0 + \tilde{b}_0). \quad (\text{A.34})$$

The conditions following from the $M = t$ gravitino equation are

$$2 \partial_i \log(Z_1 Z_2) + \partial_i \log D - D^{1/2} \left(3 (Z_1 Z_2)^{1/2} \frac{\partial_i \tilde{Z}_1}{\tilde{Z}_1^2} + (Z_1 Z_2)^{-1/2} \partial_i \tilde{Z}_2 \right) = 0, \quad (\text{A.35})$$

$$2 \partial_i f + \partial_i \tilde{b}_0 - 3 \partial_i b_0 = 0, \quad (\text{A.36})$$

and imply, together with the previous conditions,

$$\tilde{Z}_1 \tilde{Z}_2 = Z_1 Z_2, \quad \tilde{b}_0 = b_0. \quad (\text{A.37})$$

The $M = y$ gravitino equation introduces no new constraints.

In summary one has

$$\tilde{Z}_1 = Z_1, \quad \tilde{Z}_2 = Z_2, \quad \tilde{b}_0 = b_0, \quad c = f = b_0. \quad (\text{A.38})$$

In the $M = i$ components of the gravitino equation the terms in ϵ of order $1/r^4$ contribute, and one thus has to consider the possibility that the relation $\epsilon_1 = -\epsilon_2$ be violated by order $1/r^4$ terms. Hence one can write

$$\epsilon = (1 - i)\epsilon_1 + i\tilde{\epsilon}_2, \quad (\text{A.39})$$

where $\tilde{\epsilon}_2 = O(r^{-4})$. The equations one gets after taking into account the identities (A.38) are

$$\frac{D^{1/8}}{(Z_1 Z_2)^{1/4}} \left[\partial_i \epsilon_1 + \partial_i \log(Z_1^{3/16} Z_2^{1/16} Z_3^{1/4}) \epsilon_1 \right] - \frac{1}{2} \partial_i \tilde{\epsilon}_2 + \frac{1}{4} \bar{\omega}_{jk,i} \Gamma^{jk} \epsilon_1 = 0, \quad (\text{A.40})$$

and

$$\partial_i \tilde{\epsilon}_2 + \partial_i b_0 \epsilon_1 = 0. \quad (\text{A.41})$$

The second equation simply determines $\tilde{\epsilon}_2$ to be

$$\tilde{\epsilon}_2 = -b_0 \epsilon_0. \quad (\text{A.42})$$

The first equation determines ϵ_1 :

$$\epsilon_1 = Z_1^{-3/16} Z_2^{-1/16} Z_3^{-1/4} \left(1 - \frac{1}{2} b_0 \right) \epsilon_0 + \tilde{\epsilon}_1, \quad (\text{A.43})$$

with

$$\partial_i \tilde{\epsilon}_1 + \frac{1}{4} \bar{\omega}_{jk,i} \Gamma^{jk} \epsilon_0 = 0. \quad (\text{A.44})$$

The compatibility condition for the equation above is

$$\partial_{[l} \bar{\omega}_{jk,i]} \Gamma^{jk} \epsilon_0 = \bar{R}_{jk,li} \Gamma^{jk} \epsilon_0 + O(r^{-5}) = 0, \quad (\text{A.45})$$

where $\bar{R}_{ij,kl}$ is the curvature of ds_4^2 . Remembering that $\Gamma^{1234} \epsilon_0 = \epsilon_0$, the compatibility equation is equivalent to

$$\bar{R}_{ij,kl} = \frac{1}{2} \epsilon_{ijrs} \bar{R}_{rs,kl}, \quad (\text{A.46})$$

i.e. the metric ds_4^2 is hyper-Kähler.

B Closed string vertices

In this appendix we summarize our conventions for the world-sheet CFT and discuss some details of the closed string vertices used in the disk amplitudes of Section 4. The holomorphic components of the string fields satisfy the standard OPE relations

$$\begin{aligned} \partial X^\mu(z) \partial X^\nu(w) &\sim -\frac{2\alpha' \eta^{\mu\nu}}{(z-w)^2} \quad , \quad c(z) b(w) \sim \frac{1}{z-w} \quad , \\ \psi^\mu(z) \psi^\nu(w) &\sim \frac{\eta^{\mu\nu}}{z-w} \quad , \quad \gamma(z) \beta(w) \sim \frac{1}{z-w} \quad , \end{aligned} \quad (\text{B.1})$$

where b, c (β, γ) are the usual (super)ghost fields, and the full closed string coordinate x^μ is given by $x^\mu(z, \bar{z}) = (X^\mu(z) + X^\mu(\bar{z}))/2$. The simplest form for the vertex operator describing the emission of a massless NS-NS string state is

$$W_{NSNS}^{(-1,-1)} = \mathcal{G}_{\mu\nu} \psi^\mu e^{-\varphi} \tilde{\psi}^\nu e^{-\tilde{\varphi}} e^{ik \cdot x}, \quad (\text{B.2})$$

where as usual the bosonic field $\varphi(z)$ with background charge -2 and the fermionic system (η, ξ) of conformal weight $(1, 0)$ provide an equivalent description for the superghost sector

$$\gamma \simeq e^\varphi \eta \quad , \quad \beta \simeq \partial \xi e^{-\varphi} . \quad (\text{B.3})$$

The BRST charge is (we follow the conventions of [58])

$$Q_B = \oint \frac{dz}{2\pi i} \left\{ c \left(T_X + T_\psi + T_{\beta, \gamma} + (\partial c)b \right) + \gamma j_{X, \psi} - b \gamma^2 \right\} , \quad (\text{B.4})$$

where the (holomorphic parts of the) stress energy tensor and supercurrent are

$$T_X(z) = -\frac{1}{4\alpha'} \partial X^\mu \partial X_\mu , \quad T_\psi(z) = -\frac{1}{2} \psi^\mu \partial \psi_\mu , \quad (\text{B.5})$$

$$T_{\beta, \gamma}(z) = \frac{1}{2} (\partial \beta) \gamma - \frac{3}{2} \beta \partial \gamma , \quad j_{X, \psi}(z) = \frac{i}{\sqrt{2\alpha'}} \psi^\mu \partial X_\mu . \quad (\text{B.6})$$

The vertex operators (4.13) and (B.2) are invariant under separate holomorphic and antiholomorphic BRST variations, provided that we restrict to massless ($k^2 = 0$) and transverse ($k^\mu \mathcal{G}_{\mu\nu} = 0$) states.

While we can use the RR vertices (4.19) directly in the string amplitudes we are interested in, we need to discuss in more detail the NSNS vertices. First we must separate the dilaton and the graviton parts of the polarization $\mathcal{G}_{\mu\nu}$, then we must find a representative for these state whose total (holomorphic plus antiholomorphic) charge is -1 , instead of -2 .

The standard way to separate the dilaton and the graviton terms is to write the symmetric part of \mathcal{G} in two parts, with the dilaton contribution coming from the part proportional to

$$\epsilon_{\mu\nu}^{\text{dil}} = \eta_{\mu\nu} - k_\mu \ell_\nu - k_\nu \ell_\mu , \quad (\text{B.7})$$

with $\ell^2 = 0$ and $\ell_\mu k^\mu = 1$, and the graviton contribution coming from the terms orthogonal to (B.7). However this requires a non-zero value for the momentum k and requires one to choose explicitly the light-cone by fixing the null-vector ℓ . A covariant way to separate graviton and dilaton contributions is to choose different BRST representative for their vertex operators. By following [59], it is possible to show that for $k^2 = 0$ the vertex operator

$$W_{\text{dil}}^{(-2)} = \left(\eta_{\mu\nu} \psi^\mu e^{-\varphi} \tilde{\psi}^\nu e^{-\tilde{\varphi}} + c \eta \tilde{c} \partial \tilde{\xi} e^{-2\tilde{\varphi}} - c \partial \xi e^{-2\varphi} \tilde{c} \tilde{\eta} \right) e^{ik \cdot x} \quad (\text{B.8})$$

is in the BRST-cohomology of $Q_B + \tilde{Q}_B$ ⁶ and for $k \neq 0$ is equivalent to the vertex (B.2) with the dilaton polarization (B.7). Even if the vertex operator (B.8)

⁶Notice that (B.8) is not annihilated by Q_B and \tilde{Q}_B separately; also the picture of this state cannot be separated in its left and right moving part and is given by the sum of the eigenvalues of the operator $\oint \frac{dz}{2\pi i} (\xi \eta - \partial \varphi)$ and its anti-holomorphic analogue.

seems rather complicated, the state obtained via the usual operator/state correspondence is less so:

$$\lim_{z \rightarrow 0} W_{\text{dil}}^{(-2)} |0\rangle = \left(\eta_{\mu\nu} c_1 \psi_{-\frac{1}{2}}^\mu \tilde{c}_1 \tilde{\psi}_{-\frac{1}{2}}^\nu + c_1 \gamma_{-\frac{1}{2}} \tilde{c}_1 \tilde{\beta}_{-\frac{1}{2}} - c_1 \beta_{-\frac{1}{2}} \tilde{c}_1 \tilde{\gamma}_{-\frac{1}{2}} \right) |k\rangle_{-1} |\widetilde{k}\rangle_{-1} \quad (\text{B.9})$$

where $|0\rangle$ is the $SL(2, C)$ invariant vacuum (annihilated by γ_r with $r > 1/2$ and β_s with $s > -3/2$), while the states labelled with -1 are annihilated by all superghost oscillators with $r, s \geq 1/2$. Notice that the state (B.9) appears generically in the expansion of the NSNS part of the full boundary state (see for instance [60]), supporting the claim that this is the form of the dilaton state to be used in disk amplitudes without momentum flow in the Neumann directions.

For our purposes, we need to raise the picture of the vertex (B.8). Again we cannot treat the holomorphic and the anti-holomorphic part separately and thus we have to calculate $W_{NSNS}^{(-1)} = \left\{ Q_B + \tilde{Q}_B, (\xi + \tilde{\xi}) W_{NSNS}^{(-2)} \right\}$, which yields

$$\begin{aligned} W_{NSNS}^{(-1)} &= \eta_{\mu\nu} \left[(\partial X^\mu - i\alpha' k \cdot \psi \psi^\mu) \tilde{\psi}^\nu e^{-\tilde{\varphi}} + \psi^\mu e^{-\varphi} (\bar{\partial} X^\nu - i\alpha' k \cdot \tilde{\psi} \tilde{\psi}^\nu) \right] c \tilde{c} e^{ik \cdot x} \\ &\quad + \sqrt{\frac{\alpha'}{2}} \left[k \cdot \partial(\psi e^\varphi) c \tilde{\eta} \tilde{\partial} \tilde{\xi} e^{-2\tilde{\varphi}} - c \partial \xi e^{-2\varphi} k \cdot \bar{\partial}(\tilde{\psi} e^{\tilde{\varphi}}) \tilde{c} \tilde{\eta} \right] e^{ik \cdot x}. \quad (\text{B.10}) \end{aligned}$$

Let us consider what happens when this vertex is inserted in the amplitude (4.12). Because of the structure of the open string condensate discussed in Section 3.5, the only non-trivial contributions to this amplitude come from the terms in the correlator that, after the identification of the left/right moving fields, contain three ψ 's. Thus we can drop the second line as it is at most linear in ψ and focus on the terms in the first line of (B.10). The first of such terms was discussed in Section 4.2, so now we want to show that the second term, where the holomorphic part is in the -1 picture and the antiholomorphic one in the zero picture, yields exactly the same result.

The calculation of this term differs from the one discussed in Section 4.2 in two respects: first we have to identify two anti-holomorphic fermionic fields and so we clearly have contributions that are quadratic in the reflections matrix (3.6) or (3.7) (again we can use either of these two matrices, as they are identical in the $\mathbb{R}^{1,5}$ which is relevant for our purposes); then we have also to consider the non-linear nature of the bosonic boundary conditions (3.5). We will show that the extra contributions which are related to these two new features compensate each other. Actually this happens not just for the terms in the dilaton vertex (B.10), but for a generic NS-NS state in the $(-1, 0)$ picture. Thus the net result for the amplitude (4.12) obtained from these NS-NS vertices is indeed identical to the contribution obtained in Section 4.2 with the $(0, -1)$ vertices.

A first way of obtaining three ψ 's in the correlator is to start from the term $(\psi k \cdot \tilde{\psi} \tilde{\psi})$ and apply the identification (3.4) twice. By using (3.6) or (3.7), we obtain from the

reflection matrix contracted with the momentum k

$$k \cdot \tilde{\psi} = -k \cdot \psi + 2k \cdot \dot{f} \psi^v . \quad (\text{B.11})$$

For the first term, one can follow exactly the same steps discussed in Section 4.2 and obtain the result in (4.14). Notice that the -1 present in the diagonal terms R_i^i is compensated by the different ordering of the three fermionic fields. From the second term in (B.11) we get a new contribution to the mixed disk amplitude which reads

$$i\alpha' \int_0^{L_T} d\hat{v} \, 2k \cdot \dot{f} (\mathcal{G}R)_{ij} v^{ivj} e^{-ik \cdot f(\hat{v})} . \quad (\text{B.12})$$

If in this equation we take the f -independent part of the identification matrix, then this is the integral, over a full period, of the derivative of a periodic function $f(\hat{v})$. Then the only non-trivial contribution from (B.12) is from the f -dependent component R_i^u ,

$$(\text{B.12}) = -8i\alpha' \mathcal{G}_{iu} \int_0^{L_T} d\hat{v} (k \cdot \dot{f}) \dot{f}_j v^{ivj} e^{-ik \cdot f(\hat{v})} . \quad (\text{B.13})$$

Let us now focus on the contribution coming from $\bar{\partial}X^u$. So far we neglected all terms of this type because they did not give rise to correlators with the necessary three insertions of the ψ -field. However this case is different and by using the non-linear part of the identification in (3.5) we obtain

$$-8\alpha' \mathcal{G}_{iu} \int_0^{L_T} d\hat{v} \, \ddot{f}_j v^{ijv} e^{-ik \cdot f(\hat{v})} = -8i\alpha' \mathcal{G}_{iu} \int_0^{L_T} d\hat{v} (k \cdot \dot{f}) \dot{f}_j v^{ijv} e^{-ik \cdot f(\hat{v})} , \quad (\text{B.14})$$

where we integrated by parts the double derivative \ddot{f} . This result cancels (B.13) and this completes the proof of the equivalence between the vertices in $(-1, 0)$ and $(0, -1)$ pictures.

C Spinor conventions

We use the spinor conventions of [36], which we record here for completeness.

In our conventions, the 10D Majorana-Weyl spinors $\Theta_{\hat{A}}$ satisfy $\Gamma_{(10)} \Theta_{\hat{A}} = -\Theta_{\hat{A}}$, where $\Gamma_{(10)} = \Gamma_{(10)}^0 \Gamma_{(10)}^y \Gamma_{(10)}^1 \dots \Gamma_{(10)}^8$. These spinors decompose with respect to the $SO(1, 5) \times SO(4)$ as

$$\Theta_{\hat{A}} = \{\Theta_A^{\dot{\alpha}}; \Theta^{A\alpha}\} , \quad (\text{C.1})$$

where upper and lower indices $A, B, \dots = 1, \dots, 4$ denote Weyl $SO(1, 5)$ spinors of opposite chirality; similarly $\alpha, \dot{\alpha} = 1, 2$ are Weyl spinor indices of opposite chirality for the $SO(4)$ group acting along the ND T^4 directions. We decompose the 10D Gamma matrices as follows

$$\Gamma_{(10)}^a = 1_{(6)} \otimes \gamma^a, \quad \Gamma_{(10)}^I = \Gamma^I \otimes \gamma^{ND}, \quad (\text{C.2})$$

where we use simply Γ^I for the 6D Gamma matrices and

$$\begin{aligned} (\gamma^{ND})_{\dot{\alpha}}^{\beta} &= \left(\prod_a \gamma^a \right)_{\dot{\alpha}}^{\beta} = -\delta_{\dot{\alpha}}^{\beta}, \quad (\gamma^{ND})_{\alpha}^{\beta} = \left(\prod_a \gamma^a \right)_{\alpha}^{\beta} = \delta_{\alpha}^{\beta}, \\ (\Gamma)_A^B &= \left(\prod_I \Gamma^I \right)_A^B = -\delta_A^B, \quad (\Gamma)^A_B = \left(\prod_I \Gamma^I \right)^A_B = \delta^A_B. \end{aligned} \quad (\text{C.3})$$

Instead of the 6D Gamma matrices, we will often use the chiral components such as $(C\Gamma^{I_1 \dots I_{2n-1}})_{AB}$, where C is the 6D charge conjugation matrix⁷ satisfying ${}^t\Gamma^I = -C\Gamma^I C^{-1}$.

D Proof that the 4D base metric is hyper-Kahler

The form of the base metric ds_4^2 predicted by the string theory computation is

$$ds_4^2 = \left(\delta_{ij} + \bar{h}_{ij} \right) dx^i dx^j, \quad (\text{D.1})$$

with

$$\bar{h}_{ij} = (\mathbf{v}_{uli} \partial_l \mathcal{I}_j + \mathbf{v}_{ulj} \partial_l \mathcal{I}_i - \delta_{ij} \mathbf{v}_{ulk} \partial_l \mathcal{I}_k), \quad (\text{D.2})$$

where the integral \mathcal{I}_i has been defined in (5.1) and its properties are stated in (5.2).

At first order in \bar{h}_{ij} the curvature of ds_4^2 is

$$\begin{aligned} \bar{R}_{ij,kl} &= \frac{1}{2} \left(\partial_k \partial_j \bar{h}_{il} - \partial_l \partial_j \bar{h}_{ik} - (i \leftrightarrow j) \right) \\ &\equiv \left((1)_{ij,kl} + (2)_{ij,kl} - (3)_{ij,kl} - (i \leftrightarrow j) \right), \end{aligned} \quad (\text{D.3})$$

where

$$(1)_{ij,kl} \equiv \mathbf{v}_{umi} \partial_k \partial_j \partial_m \mathcal{I}_l - (k \leftrightarrow l), \quad (\text{D.4})$$

$$(2)_{ij,kl} \equiv \mathbf{v}_{uml} \partial_k \partial_j \partial_m \mathcal{I}_i - (k \leftrightarrow l), \quad (\text{D.5})$$

$$(3)_{ij,kl} \equiv \delta_{il} \mathbf{v}_{umn} \partial_k \partial_j \partial_m \mathcal{I}_n - (k \leftrightarrow l). \quad (\text{D.6})$$

We want to compute $\frac{1}{2} \epsilon_{klpq} \bar{R}_{ij,pq}$ and prove that it is equal to $\bar{R}_{ij,kl}$. We will do it term by term:

⁷ C is related to the 10D and 4D charge conjugation matrices by $C_{10} = C \otimes C_4$.

(1):

$$\frac{1}{2} \epsilon_{klpq} (1)_{ij,pq} = \epsilon_{klpq} \mathbf{v}_{umi} \partial_p \partial_j \partial_m \mathcal{I}_q = -\frac{1}{2} \epsilon_{klpq} \epsilon_{mirs} \mathbf{v}_{urs} \partial_p \partial_j \partial_m \mathcal{I}_q, \quad (\text{D.7})$$

where we have used, in the second equality, the anti-self-duality of \mathbf{v}_{uij} . The product of the two epsilon's gives, up to the exchange of r and s that cancels the factor $1/2$, 12 possible terms: the 3 terms proportional to δ_{pm} and the 3 terms proportional to δ_{qm} vanish thanks to (5.2); the 2 terms proportional to δ_{pi} are symmetric in i, j and hence cancel after anti-symmetrization in i, j ; the 4 terms left give

$$\begin{aligned} \frac{1}{2} \epsilon_{klpq} (1)_{ij,pq} &= \left(\mathbf{v}_{upl} \partial_p \partial_j \partial_k \mathcal{I}_i - (k \leftrightarrow l) \right) - \left(\delta_{il} \mathbf{v}_{upq} \partial_p \partial_j \partial_k \mathcal{I}_q - (k \leftrightarrow l) \right) \\ &= (2)_{ij,kl} - (3)_{ij,kl}, \end{aligned} \quad (\text{D.8})$$

where the second equality follows from the first property in (5.2).

(2):

$$\frac{1}{2} \epsilon_{klpq} (2)_{ij,pq} = \epsilon_{klpq} \mathbf{v}_{umq} \partial_p \partial_j \partial_m \mathcal{I}_i = -\frac{1}{2} \epsilon_{klpq} \epsilon_{mqr s} \mathbf{v}_{urs} \partial_p \partial_j \partial_m \mathcal{I}_i; \quad (\text{D.9})$$

the contraction of the two epsilon's produces, up to r and s exchange, 3 different terms: the term with δ_{pm} vanishes due to (5.2), and the other 2 terms give

$$\frac{1}{2} \epsilon_{klpq} (2)_{ij,pq} = \mathbf{v}_{upl} \partial_p \partial_j \partial_k \mathcal{I}_i = (2)_{ij,kl}. \quad (\text{D.10})$$

(3):

$$\frac{1}{2} \epsilon_{klpq} (3)_{ij,pq} = \epsilon_{klpq} \delta_{iq} \mathbf{v}_{umn} \partial_p \partial_j \partial_m \mathcal{I}_n = -\frac{1}{2} \epsilon_{klpq} \epsilon_{mnrs} \delta_{iq} \mathbf{v}_{urs} \partial_p \partial_j \partial_m \mathcal{I}_n; \quad (\text{D.11})$$

for the reasons explained above, of the 12 terms coming from the expansion of the two uncontracted epsilon's the ones containing δ_{pm} , δ_{im} or δ_{pn} vanish, leaving

$$\begin{aligned} \frac{1}{2} \epsilon_{klpq} (3)_{ij,pq} &= -\left(\mathbf{v}_{upi} \partial_p \partial_j \partial_k \mathcal{I}_l - (k \leftrightarrow l) \right) + \left(\mathbf{v}_{upl} \partial_p \partial_j \partial_k \mathcal{I}_i - (k \leftrightarrow l) \right) \\ &= -(1)_{ij,kl} + (2)_{ij,kl}. \end{aligned} \quad (\text{D.12})$$

Putting things together:

$$\begin{aligned} \frac{1}{2} \epsilon_{klpq} \bar{R}_{ij,pq} &= \left((2)_{ij,kl} - (3)_{ij,kl} + (2)_{ij,kl} + (1)_{ji,kl} - (2)_{ji,kl} - (i \leftrightarrow j) \right) \\ &= \bar{R}_{ij,kl}, \end{aligned} \quad (\text{D.13})$$

which is the identity we wanted to prove.

References

- [1] A. Strominger and C. Vafa, “Microscopic origin of the Bekenstein-Hawking entropy,” *Phys.Lett.* **B379** (1996) 99–104, [hep-th/9601029](#).
- [2] C. G. Callan and J. M. Maldacena, “D-brane approach to black hole quantum mechanics,” *Nucl.Phys.* **B472** (1996) 591–610, [hep-th/9602043](#).
- [3] S. D. Mathur, “The fuzzball proposal for black holes: An elementary review,” *Fortsch. Phys.* **53** (2005) 793–827, [hep-th/0502050](#).
- [4] S. D. Mathur, “Fuzzballs and the information paradox: a summary and conjectures,” [0810.4525](#).
- [5] K. Skenderis and M. Taylor, “The fuzzball proposal for black holes,” *Phys. Rept.* **467** (2008) 117–171, [0804.0552](#).
- [6] V. Balasubramanian, J. de Boer, S. El-Showk, and I. Messamah, “Black Holes as Effective Geometries,” *Class.Quant.Grav.* **25** (2008) 214004, [0811.0263](#).
- [7] B. D. Chowdhury and A. Virmani, “Modave Lectures on Fuzzballs and Emission from the D1-D5 System,” [1001.1444](#).
- [8] S. W. Hawking, “Particle Creation by Black Holes,” *Commun. Math. Phys.* **43** (1975) 199–220.
- [9] S. W. Hawking, “Breakdown of Predictability in Gravitational Collapse,” *Phys. Rev.* **D14** (1976) 2460–2473.
- [10] S. D. Mathur, “The information paradox: A pedagogical introduction,” *Class. Quant. Grav.* **26** (2009) 224001, [0909.1038](#).
- [11] S. D. Mathur and C. J. Plumberg, “Correlations in Hawking radiation and the infall problem,” [1101.4899](#).
- [12] S. D. Mathur, “What the information paradox is *not*,” [1108.0302](#).
- [13] I. Bena and N. P. Warner, “Black holes, black rings and their microstates,” *Lect. Notes Phys.* **755** (2008) 1–92, [hep-th/0701216](#).
- [14] A. Dabholkar, J. P. Gauntlett, J. A. Harvey, and D. Waldram, “Strings as Solitons & Black Holes as Strings,” *Nucl. Phys.* **B474** (1996) 85–121, [hep-th/9511053](#).
- [15] C. G. Callan, J. M. Maldacena, and A. W. Peet, “Extremal Black Holes As

- Fundamental Strings,” *Nucl. Phys.* **B475** (1996) 645–678, [hep-th/9510134](#).
- [16] O. Lunin and S. D. Mathur, “Metric of the multiply wound rotating string,” *Nucl. Phys.* **B610** (2001) 49–76, [hep-th/0105136](#).
 - [17] O. Lunin and S. D. Mathur, “AdS/CFT duality and the black hole information paradox,” *Nucl. Phys.* **B623** (2002) 342–394, [hep-th/0109154](#).
 - [18] M. Taylor, “General 2 charge geometries,” *JHEP* **03** (2006) 009, [hep-th/0507223](#).
 - [19] I. Kanitscheider, K. Skenderis, and M. Taylor, “Fuzzballs with internal excitations,” *JHEP* **06** (2007) 056, [0704.0690](#).
 - [20] I. Bena and N. P. Warner, “One ring to rule them all ... and in the darkness bind them?,” *Adv.Theor.Math.Phys.* **9** (2005) 667–701, [hep-th/0408106](#).
 - [21] I. Bena and N. P. Warner, “Bubbling supertubes and foaming black holes,” *Phys.Rev.* **D74** (2006) 066001, [hep-th/0505166](#).
 - [22] P. Berglund, E. G. Gimon, and T. S. Levi, “Supergravity microstates for BPS black holes and black rings,” *JHEP* **0606** (2006) 007, [hep-th/0505167](#).
 - [23] I. Bena, C.-W. Wang, and N. P. Warner, “The Foaming three-charge black hole,” *Phys.Rev.* **D75** (2007) 124026, [hep-th/0604110](#).
 - [24] I. Bena, C.-W. Wang, and N. P. Warner, “Plumbing the Abyss: Black ring microstates,” *JHEP* **0807** (2008) 019, [0706.3786](#).
 - [25] V. Balasubramanian, E. G. Gimon, and T. S. Levi, “Four Dimensional Black Hole Microstates: From D-branes to Spacetime Foam,” *JHEP* **0801** (2008) 056, [hep-th/0606118](#).
 - [26] J. de Boer, S. El-Showk, I. Messamah, and D. Van den Bleeken, “Quantizing N=2 Multicenter Solutions,” *JHEP* **0905** (2009) 002, [0807.4556](#).
 - [27] I. Bena, N. Bobev, S. Giusto, C. Ruef, and N. P. Warner, “An Infinite-Dimensional Family of Black-Hole Microstate Geometries,” *JHEP* **1103** (2011) 022, [1006.3497](#).
 - [28] J. de Boer, S. El-Showk, I. Messamah, and D. Van den Bleeken, “A Bound on the entropy of supergravity?,” *JHEP* **1002** (2010) 062, [0906.0011](#).
 - [29] K. Skenderis and M. Taylor, “Anatomy of bubbling solutions,” *JHEP* **0709** (2007) 019, [0706.0216](#).

- [30] S. Giusto, S. D. Mathur, and A. Saxena, “Dual geometries for a set of 3-charge microstates,” *Nucl. Phys.* **B701** (2004) 357–379, [hep-th/0405017](#).
- [31] S. Giusto and S. D. Mathur, “Geometry of D1-D5-P bound states,” *Nucl. Phys.* **B729** (2005) 203–220, [hep-th/0409067](#).
- [32] J. Ford, S. Giusto, and A. Saxena, “A class of BPS time-dependent 3-charge microstates from spectral flow,” *Nucl. Phys.* **B790** (2008) 258–280, [hep-th/0612227](#).
- [33] I. Bena, B. D. Chowdhury, J. de Boer, S. El-Showk, and M. Shigemori, “Moulting Black Holes,” [1108.0411](#).
- [34] P. Di Vecchia *et al.*, “Classical p-branes from boundary state,” *Nucl. Phys.* **B507** (1997) 259–276, [hep-th/9707068](#).
- [35] P. Di Vecchia, M. Frau, A. Lerda, and A. Liccardo, “(F,Dp) bound states from the boundary state,” *Nucl. Phys.* **B565** (2000) 397–426, [hep-th/9906214](#).
- [36] S. Giusto, J. F. Morales, and R. Russo, “D1D5 microstate geometries from string amplitudes,” *JHEP* **03** (2010) 130, [0912.2270](#).
- [37] S. R. Das and S. D. Mathur, “Excitations of D strings, entropy and duality,” *Phys.Lett.* **B375** (1996) 103–110, [hep-th/9601152](#).
- [38] Y. Hikida, H. Takayanagi, and T. Takayanagi, “Boundary states for D-branes with traveling waves,” *JHEP* **04** (2003) 032, [hep-th/0303214](#).
- [39] J. D. Blum, “Gravitational radiation from travelling waves on D- strings,” *Phys. Rev.* **D68** (2003) 086003, [hep-th/0304173](#).
- [40] C. P. Bachas and M. R. Gaberdiel, “World-sheet duality for D-branes with travelling waves,” *JHEP* **03** (2004) 015, [hep-th/0310017](#).
- [41] W. Black, R. Russo, and D. Turton, “The supergravity fields for a D-brane with a travelling wave from string amplitudes,” *Phys. Lett.* **B694** (2010) 246–251, [1007.2856](#).
- [42] V. Gurarie, “Logarithmic operators in conformal field theory,” *Nucl. Phys.* **B410** (1993) 535–549, [hep-th/9303160](#).
- [43] V. Periwal and O. Tafjord, “D-brane recoil,” *Phys. Rev.* **D54** (1996) 3690–3692, [hep-th/9603156](#).
- [44] I. I. Kogan, N. E. Mavromatos, and J. F. Wheeler, “D-brane recoil and logarithmic operators,” *Phys. Lett.* **B387** (1996) 483–491, [hep-th/9606102](#).

- [45] I. I. Kogan and J. F. Wheeler, “Boundary logarithmic conformal field theory,” *Phys. Lett.* **B486** (2000) 353–361, [hep-th/0003184](#).
- [46] N. D. Lambert, H. Liu, and J. M. Maldacena, “Closed strings from decaying D-branes,” *JHEP* **03** (2007) 014, [hep-th/0303139](#).
- [47] I. Bena, J. de Boer, M. Shigemori, and N. P. Warner, “Double, Double Supertube Bubble,” [1107.2650](#).
- [48] V. Balasubramanian, J. de Boer, V. Jejjala, and J. Simon, “The library of Babel: On the origin of gravitational thermodynamics,” *JHEP* **12** (2005) 006, [hep-th/0508023](#).
- [49] V. Balasubramanian, B. Czech, K. Larjo, and J. Simon, “Integrability vs. information loss: A simple example,” *JHEP* **11** (2006) 001, [hep-th/0602263](#).
- [50] V. Balasubramanian, B. Czech, K. Larjo, D. Marolf, and J. Simon, “Quantum geometry and gravitational entropy,” *JHEP* **12** (2007) 067, [0705.4431](#).
- [51] C. G. Callan, Jr., C. Lovelace, C. R. Nappi, and S. A. Yost, “Loop Corrections to Superstring Equations of Motion,” *Nucl. Phys.* **B308** (1988) 221.
- [52] C. Bachas, “Relativistic string in a pulse,” *Ann. Phys.* **305** (2003) 286–309, [hep-th/0212217](#).
- [53] M. Bertolini, M. Billo, A. Lerda, J. F. Morales, and R. Russo, “Brane world effective actions for D-branes with fluxes,” *Nucl. Phys.* **B743** (2006) 1–40, [hep-th/0512067](#).
- [54] H. Liu, G. W. Moore, and N. Seiberg, “Strings in a time-dependent orbifold,” *JHEP* **06** (2002) 045, [hep-th/0204168](#).
- [55] H. Liu, G. W. Moore, and N. Seiberg, “Strings in time-dependent orbifolds,” *JHEP* **10** (2002) 031, [hep-th/0206182](#).
- [56] M. Billo *et al.*, “Classical gauge instantons from open strings,” *JHEP* **02** (2003) 045, [hep-th/0211250](#).
- [57] M. Billo, M. Frau, L. Giaccone, and A. Lerda, “Holographic non-perturbative corrections to gauge couplings,” *JHEP* **1108** (2011) 007, [1105.1869](#).
- [58] J. Polchinski, *String theory. Vol. 2: Superstring Theory and Beyond*. Cambridge Univ. Pr., 1998.
- [59] O. Bergman and B. Zwiebach, “The Dilaton theorem and closed string backgrounds,” *Nucl. Phys.* **B441** (1995) 76–118, [hep-th/9411047](#).

- [60] M. Billo *et al.*, “Microscopic string analysis of the D0-D8 brane system and dual R-R states,” *Nucl. Phys.* **B526** (1998) 199–228, [hep-th/9802088](#).